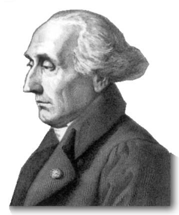
Rigid Bodies

Monday, 11 November 2013

Physics 111



Changes in angular momentum are caused by external torques. While the momentum of a massive particle is aligned with its velocity, the angular momentum of a rotating rigid body typically is not aligned with its angular velocity. Even in the absence of external torques, this means that the angular velocity vector precesses about the angular momentum vector.

Perhaps it is wise to recap swiftly the basics from Physics 24.

1. A force $\mathbf{F}$ applied at a position $\mathbf{r}$ with respect to a designated origin produces a torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ about that origin.
2. The angular momentum of a collection of point particles about a designated origin is $\mathbf{L} = \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha$, where the prime indicates measurement with respect to the designated origin.
3. The angular momentum may be expressed as the sum of two parts: an orbital part ($\mathbf{R} \times \mathbf{P}$) expressed in terms of the center-of-mass position and momentum, and a spin part ($\sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha$) expressed in terms of coordinates measured with respect to the center of mass.
4. In the absence of a net external torque on a system, its (total) angular momentum in an inertial frame is conserved. Choosing an origin wisely can often simplify the analysis of a problem.
5. The rotational kinetic energy of a spinning rigid body may be expressed as $T = \frac{1}{2} I\omega^2$, where I is the moment of inertia for rotation about the spin axis.
6. The moment of inertia for rotation about an axis is given by the sum $I = \sum_\alpha m_\alpha r_\alpha^2$, where r_α is the perpendicular distance from the axis to mass point α.

The last two points have been highlighted in red because they will need to be modified here to handle situations in which the axis of rotation itself changes orientation during the motion or in which the instantaneous axis of rotation does not coincide with an axis of symmetry of the object.

We will not have time for an exhaustive treatment of rotating rigid bodies; their motions can be quite complicated. We will, however, develop the correct equations of motion and analyze a few cases of interest and importance.

1. The Angular Velocity and Angular Momentum Aren't Necessarily Parallel

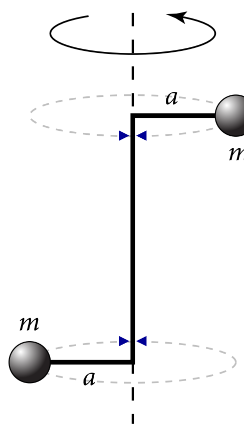


Figure 1: *An asymmetric rotor rotates about a vertical axis. Describe the forces that must be supplied by the bearings (shown in blue) to keep the rotor rotating about a vertical axis. (These include upward forces to keep it from falling.)*

You might think that when you spin an object about an axis, its angular momentum would point along that axis, parallel to its angular velocity. It always seemed to work that way in Physics 24. To see why life isn't always so easy, consider the asymmetric rotor illustrated in Fig. 1, which is made to spin at steady angular velocity about the vertical axis. Imagine that each mass were tied to a string. For the mass to go around in a circle, the string must supply an inward (centripetal) force; without it, the mass would proceed in a straight line, not around a circle. As the rotor spins about the vertical axis, the support bearings (shown in blue triangles) must supply various forces to keep the axis vertical. How do those forces vary in time? Do they supply a torque to the rotor? If so, what does this imply about the rotor's angular momentum?

The asymmetric rotor illustrates the perhaps surprising fact that the angular velocity need not be parallel to the angular momentum, and that a rigid body cannot spin placidly about just any axis we please. In the absence of external torques, a system's angular momentum is conserved, of course. Unlike conservation of total momentum, however, the conservation of angular momentum can give rise to some remarkably complicated and counterintuitive motions of rigid bodies.

As the asymmetric rotor shows, the angular momentum and the angular velocity of a rotating object may not be parallel. Think of a wobbling football, for example.

As with much of "Newtonian" mechanics, fundamental advances in describing the rotation of rigid bodies were made by the Swiss mathematician and physicist, Leonhard Euler (1707–1783). Euler found general expressions for the kinetic energy and angular momentum of a rigid body, in terms of the moment of inertia tensor, and derived the equations of motion of symmetric and asymmetric tops. We will derive Euler's equations for a torque-free top and use them to explain the curious behavior of tossed tennis rackets.

2. Kinetic Energy of a Rigid Body

We showed that the kinetic energy of a system of mass points m_α may be expressed as the sum of two terms: $\frac{1}{2}MV^2$ and $\frac{1}{2}\sum_\alpha m_\alpha \dot{r}_\alpha^2$, where $M = \sum_\alpha m_\alpha$ is the total mass, $\mathbf{V}$ is the velocity of the center of mass, and $\mathbf{r}_\alpha$ is the position of particle α with respect to the center of mass (the vector points from the center of mass to particle α). We now apply this for the special case that the mass points compose a rigid body. In this case, the two parts of the kinetic energy may be expressed

$$T_{\text{trans}} = \frac{1}{2}\sum_\alpha m_\alpha V^2 = \frac{1}{2}MV^2 \tag{1}$$

$$T_{\text{rot}} = \frac{1}{2}\sum_\alpha m_\alpha |\boldsymbol{\omega}\times\mathbf{r}_\alpha|^2 \tag{2}$$

Define the Levi-Civita ε symbol[1] by

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk = 123 = xyz \text{ or cyclic permutations} \\ -1 & ijk = 132 = xzy \text{ or cyclic permutations} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

so that, using the Einstein summation convention, we can write

$$(\mathbf{A}\times\mathbf{B})_i = \varepsilon_{ijk}A_jB_k \tag{4}$$

We now use an important identity for the Levi-Civita ε symbol,

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \tag{5}$$

which is proved at the end of these notes. Remember, we are using the summation convention, so the repeated index i is summed over. Using these definitions and results, we can derive

$$\begin{aligned} |\mathbf{A}\times\mathbf{B}|^2 &= (\varepsilon_{ijk}A_jB_k)(\varepsilon_{ilm}A_lB_m) = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl})A_jB_kA_lB_m \\ &= A_iA_iB_jB_j - A_iB_iA_jB_j = A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2 \end{aligned} \tag{6}$$

This expression allows us to rewrite Eq. (2) as

$$T_{\text{rot}} = \frac{1}{2}\sum_\alpha m_\alpha(\omega_i\omega_i r_{\alpha j}r_{\alpha j} - \omega_i\omega_j r_{\alpha i}r_{\alpha j}) = \frac{1}{2}\omega_i\omega_j\sum_\alpha m_\alpha(r_\alpha^2\delta_{ij} - r_{\alpha i}r_{\alpha j}) \tag{7}$$

[1] Tullio Levi-Civita (1873–1941) was an Italian mathematician whose 1900 book with Ricci-Curbastro on the theory of tensors, *Méthodes de calcul différentiel absolu et leurs applications*, was used by Einstein to learn the tensor calculus. When asked much later what he liked best about Italy, Einstein said "spaghetti and Levi-Civita." Levi-Civita corresponded with Einstein, worked on general relativity and later on quantum mechanics and Dirac's equation. Because he was Jewish, Levi-Civita was stripped of his professorship in 1938 and died in Rome in 1941.

We define the sum in this expression to be the **inertia tensor** (or moment of inertia tensor) I_{ij},

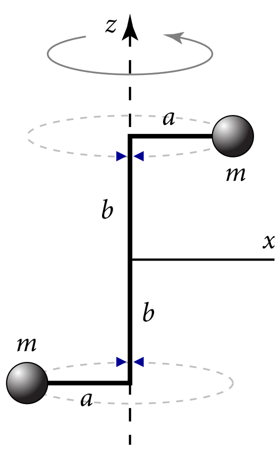
$$\boxed{I_{ij} = \sum_\alpha m_\alpha (r_\alpha^2 \delta_{ij} - r_{\alpha i} r_{\alpha j})} \tag{8}$$

in terms of which we may write the kinetic energy as

$$\boxed{T_{\text{rot}} = \frac{1}{2} I_{ij} \omega_i \omega_j} \tag{9}$$

In Eq. (8), $\mathbf{r}_\alpha$ is the position vector of particle α with respect to the center of mass, r_α is the magnitude of the position vector (the distance from the center of mass), and $r_{\alpha i}$ is the ith component of $\mathbf{r}_\alpha$.

Exercise 1 Compute the components of the inertia tensor I_{ij} for the asymmetric rotor shown here, assuming that the mass of the connecting rods is negligible compared to the two spheres. Neglect the spatial extent of the two masses.



3. Angular Momentum

Just as the kinetic energy of a system of mass points may be expressed as the sum of the translational kinetic energy of a mass point $M = \sum_\alpha m_\alpha$ at the center of mass and the kinetic energy of each constituent mass point *with respect to the center of mass*, the angular momentum about any chosen origin may be similarly factored:

$$\mathbf{L} \equiv \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha = \underbrace{\mathbf{R} \times \mathbf{P}}_{\text{orbital}} + \underbrace{\sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha}_{\text{spin}} \tag{10}$$

In this expression, **R** is the position of the center of mass with respect to the chosen origin, and **P** is the total momentum of the particles. The final summation involves quantities measured with respect to the center of mass.

Exercise 2 Prove Eq. (10).

We now seek to evaluate the spin angular momentum on the right-hand side of Eq. (10) for a rigid body, for which $\dot{\mathbf{r}}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha$. Therefore,

$$\mathbf{L}_{\text{spin}} = \sum_\alpha \mathbf{r}_\alpha \times \mathbf{p}_\alpha = \sum_\alpha m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)$$

Once again, employ the Levi-Civita symbol to simplify this expression to show that $L_i = I_{ij}\omega_j$.

That is, the (spin) angular momentum is given by the dot product of the moment of inertia tensor and the angular velocity,

$$L_i = I_{ij}\omega_j \qquad \text{or} \qquad \mathbf{L} = \overleftrightarrow{\mathbf{I}} \cdot \boldsymbol{\omega} \tag{11}$$

Comparing to Eq. (9), we see that we may also represent the rotational kinetic energy as

$$\boxed{T_{\text{rot}} = \frac{1}{2}\omega_i L_i = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}} \tag{12}$$

4. Tensors

What's a tensor? A simple answer is that a tensor is an object that eats a vector and returns a vector. It is linear: if you give it an input vector twice as long, you get out an output vector twice as long as the original output vector and pointing in the same direction. *It is the most general linear beast that maps vectors into other vectors that need not point in the same direction.*

A tensor maps vectors into other vectors that need not point in the same direction such that doubling the input doubles the output.

Another answer is that a tensor is a generalization of a vector. In fact a vector is a first-rank tensor. So, what's a **vector**? A vector has a magnitude and a direction. It may be represented in a particular coordinate system by projecting onto an orthogonal set of basis unit vectors, $A_i = \mathbf{A}\cdot\mathbf{e}_i$, but the vector has a life independent of any particular coordinate representation. Rotating the coordinate axes changes the unit vectors $\mathbf{e}_i$ and therefore the components of the vector, A_i, in the rotated coordinate system, but does not change the vector—only its representation. This is the *passive* view of rotation.

Alternatively, rotating the vector in the opposite direction while keeping the coordinate system fixed produces the same change in the coordinate representation of the vector (the components A_i) as the passive rotation. This is called an *active* rotation. Some authors prefer to focus on active rotations; others on passive rotations. *In either case, a quantity is a vector if it behaves under rotations as a vector does.*

And how exactly does a vector behave? Rotations preserve length ($A_i A_i = A_{j'} A_{j'}$), and they preserve the handedness of the coordinate system. In summation notation, the components of the vector **A** in the rotated (primed) coordinate system are given in terms of the unrotated (unprimed) system by

$$A_{i'} = \lambda_{i'j} A_j \tag{13}$$

where $\lambda_{i'j} = \cos(x_{i'}, x_j) = \mathbf{e}_{i'} \cdot \mathbf{e}_j$. The direction cosine between two directions is the cosine of the angle between them. This definition of the direction cosine makes it clear that $\lambda_{ij'} = \lambda_{j'i}$.

Furthermore, the nature of the inverse transformation is clear: rotate by the same angle in the opposite direction. The transformation matrix thus satisfies the **orthogonality condition**

$$\lambda_{ij}\lambda_{kj} = \delta_{ik} = \lambda_{ji}\lambda_{jk} \tag{14}$$

How does a tensor behave? The inertia tensor for a single mass point is $I_{ij} = m(r^2\delta_{ij} - r_i r_j)$, which contains two vector components. On rotating the coordinate system, we need to rotate each component. The tensor analog to the transformation equation for vectors, Eq. (13), is thus

$$B_{i'j'} = \lambda_{i'k}\lambda_{j'l}B_{kl} \tag{15}$$

That is, we need one rotation matrix λ for each tensor index. It is straightforward to generalize this expression to tensors of arbitrary rank.

Because the number of tensor indices can be greater than two, it is helpful not to tie too closely in your mind the idea of a tensor with that of a square matrix. A second-rank tensor, such as I_{ij}, can certainly be represented by a three-by-three matrix, once a set of basis vectors has been selected. But just as a vector has a life independent of the basis one chooses, so does a tensor.

5. Inertia Tensor

The inertia tensor defined by Eq. (8) has an important property that greatly simplifies the description of rotating rigid bodies: we can always find a body coordinate system that diagonalizes the inertia tensor. In that coordinate system, the only nonzero elements of I_{ij} are those with $i = j$. This means that if the body rotates about one of these **principal axes**, the angular velocity and the angular momentum are aligned. The proof follows closely the discussion of linear algebra in the notes on coupled oscillators. The **principal moments of inertia** are obtained by setting

$$|I_{ij} - I\delta_{ij}| = 0 \tag{16}$$

where I_{ij} is the inertia tensor in any Cartesian coordinate system. The principle moments are always real and positive, and the principal axes are automatically

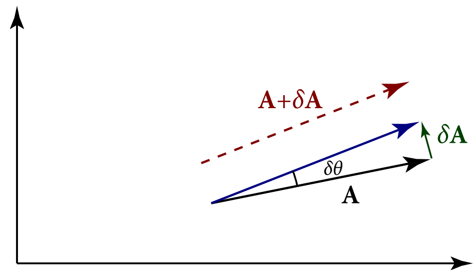


Figure 2: *The frame in which* **A** *is at rest rotates through angle* $\delta\theta$*, "displacing"* **A** *to* **A**+δ**A** *(red). Translating* **A** *back to align the tail of the vector to the original vector* **A***, allows us to identify* δ**A***, the change in* **A** *(green). Note that the vector direction of* $\delta\boldsymbol{\theta}$ *is perpendicular to the plane of the figure and determined by the right-hand rule.*

orthogonal if the three principal moments are distinct, or may be chosen to be orthogonal if two or more moments are identical.[2] Therefore, we may choose to use the principal axes to express the (spin) angular momentum (about the center of mass) as

$$L_i = I_i \omega_i \qquad \text{(no sum)} \tag{17}$$

where $I_i \equiv I_{ii}$.

It is always possible to find an orthogonal set of axes (in the body frame) such that the inertia tensor has nonzero values only along the main diagonal.

6. Equations of Motion

We can develop the equations of motion for a rotating object in a couple of different ways, one using the approach below, and the other using the "Eulerian angles" as generalized coordinates to express the angular orientation of the rotating body with respect to an inertial coordinate system. I'll leave the Eulerian angles for another day.

The first step in the derivation is to consider how a vector "in a rotating frame" changes with time in an inertial frame. Consider a vector **A** at rest in a frame that rotates *with respect to an inertial frame* through an infinitesimal angle $\delta\boldsymbol{\theta}$. As shown in Fig. 2, the change in **A** is perpendicular to both **A** and $\delta\boldsymbol{\theta}$ and has magnitude $\delta\theta\, A$. If **A** had a component parallel to $\delta\boldsymbol{\theta}$, that component would not be modified by the rotation; it is only the component in the plane perpendicular to the rotation axis that counts. The upshot is that the change in **A** is given by

$$\delta\mathbf{A} = \delta\boldsymbol{\theta} \times \mathbf{A}$$

If the rotation takes place in time δt, then taking the limit as $\delta t \to 0$ we obtain

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\text{in}} = \boldsymbol{\omega} \times \mathbf{A} \tag{18}$$

[2] In Physics 116 you will see a more general result that **Hermitian** matrices, which satisfy $\mathsf{H}_{ji} = (\mathsf{H}_{ij})^*$, have real eigenvalues.

(The subscript "in" refers to the inertial frame.) The total time derivative on the left gives the change in the vector **A** in the inertial frame, the angular velocity vector $\boldsymbol{\omega}$ indicates the rotation of the body (rotating) frame *with respect to the inertial frame*, and **A** on the right is the vector in the rotating frame.

Don't fall into the trap of thinking about $\boldsymbol{\omega}$ as zero in the rotating frame, since the rotating frame is not rotating with respect to itself. That's not how the angular velocity is defined. It describes the rotation of the body frame **with respect to an inertial frame**.

If, in addition, the vector **A** itself is changing in the rotating frame, then the change in **A** in the inertial frame is

$$\boxed{\left(\frac{d\mathbf{A}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\mathbf{A}} \tag{19}$$

Equation (19) describes how to relate the change in any vector when expressed in a rotating frame to its change in an inertial frame.

Of course, the thing that changes angular momentum is torque:

$$\boldsymbol{\tau} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{in}} \tag{20}$$

The problem, however, is that it is much easier to express the angular momentum in a body-centered coordinate system, using Eq. (17). Using Eq. (19), then, we can express the rate of change of angular momentum in the inertial (lab) frame as

$$\left(\frac{d\mathbf{L}}{dt}\right)_{\text{in}} = \left(\frac{d\mathbf{L}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega}\times\mathbf{L} \tag{21}$$

In the rotating frame, the principal moments of inertia do not change with time, so

$$\left(\frac{dL_j}{dt}\right)_{\text{rot}} = I_j\dot{\omega}_j \qquad \text{(no sum)}$$

The cross product term may be expressed

$$(\boldsymbol{\omega}\times\mathbf{L})_i = \varepsilon_{ijk}\omega_j L_k = \varepsilon_{ijk}\omega_j I_k\omega_k$$

which can perhaps be more readily understood by picking a single component (say, the body x component) to give

$$(\boldsymbol{\omega}\times\mathbf{L})_1 = \omega_2\omega_3(I_3 - I_2)$$

and cyclic permutations. So, combining Eq. (20) and Eq. (21), we get

$$\begin{aligned}
\tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\
\tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\
\tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2
\end{aligned} \tag{22}$$

which are Euler's equations.

If we consider the motion of a rotating object that is not subject to an external torque, then these equations simplify to the set of coupled nonlinear first-order differential equations

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3 \qquad \dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 \qquad \dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2$$

Exercise 3 – Symmetric top Suppose that a rigid body has enough symmetry that $I_1 = I_2$, but $I_1 \neq I_3$. Such an object is called a symmetric top. Show that the angular velocity vector precesses around the $\mathbf{e}_3$ axis and determine the rate of precession.

6.1 Tennis Racket

We can now understand the behavior of the thrown tennis racket. In flight, its center of mass falls with constant acceleration g, but no external torque acts (if we may neglect air resistance). The racket's principal axes are along the handle (moment I_1), perpendicular to the handle and parallel to the face (moment I_2), and perpendicular to both the handle and the face (moment I_3). We will toss the racket attempting to spin it about one of the principal axes—say x_1—but will inevitably impart some small angular velocity about the other two principal axes. If the rotation is stable, the magnitude of the angular velocity about the x_2 and x_3 axes will oscillate and remain small; if it is unstable, their magnitudes will grow exponentially.

We will take as the initial angular velocity,

$$\boldsymbol{\omega} = \omega_1\,\mathbf{e}_1 + \delta_2\,\mathbf{e}_2 + \delta_3\,\mathbf{e}_3$$

where $|\delta_2| \ll \omega_1$ and $|\delta_3| \ll \omega_1$. From Eq. (22),

$$\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\omega_3$$
$$\dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\omega_1\right)\omega_2$$

Differentiating the first equation (treating ω_1 as a constant), and using the second equation to replace $\dot{\omega}_3$, we get

$$\ddot{\omega}_2 \approx \left(\frac{I_3 - I_1}{I_2}\omega_1\right)\left(\frac{I_1 - I_2}{I_3}\omega_1\right)\omega_2 = \left(\frac{(I_3 - I_1)(I_1 - I_2)}{I_2 I_3}(\omega_1)^2\right)\omega_2$$

We can also differentiate the second and use the first to eliminate $\dot{\omega}_2$. Let

$$(\Omega_1)^2 = \frac{(I_3 - I_1)(I_2 - I_1)}{I_2 I_3}(\omega_1)^2$$

Then

$$\ddot{\omega}_2 + (\Omega_1)^2\omega_2 = 0 \qquad \text{and} \qquad \ddot{\omega}_3 + (\Omega_1)^2\omega_3 = 0$$

As defined above, $I_1 < I_2 < I_3$, so $(\Omega_1)^2 > 0$ and the solutions for $\omega_2(t)$ and $\omega_3(t)$ are both oscillatory. Hence, rotation about the principal axis with the smallest moment of inertia is stable. Of course, we can cyclically permute the indices to investigate stability for rotation about the other two axes, as well. We find that rotation about x_3 is stable, but when we seek to spin about the axis with the intermediate moment of inertia,

$$(\Omega_2)^2 = \frac{(I_1 - I_2)(I_3 - I_2)}{I_3 I_1}(\omega_2)^2 < 0$$

The angular velocity components ω_1 and ω_3 depend exponentially on the time, and so motion about x_2 is unstable.

7. Summary

- The kinetic energy of a system of mass points may be decomposed as the sum of the center of mass kinetic energy (all the mass concentrated at the center of mass and moving with the center-of-mass velocity) and the kinetic energy with respect to the center of mass. For a rigid body, all the particles undergo a coordinated motion with respect to the center of mass, allowing us to simplify the expression for the kinetic energy.

- Finite rotations do not commute, but infinitesimal rotations do commute. Hence, angular velocity is a vector.
- In general, the angular momentum of a rigid body is not aligned with its angular velocity. Strange, but true. Nonetheless, the angular momentum (about the center of mass) *is* proportional to the angular velocity, $L_i = I_{ij}\omega_j$. The quantity $I_{ij} = \sum_\alpha m_\alpha(r_\alpha^2\delta_{ij} - r_{\alpha i}r_{\alpha j})$ is called the inertia tensor.
- Because it is a real, symmetric second-rank tensor, it is always possible to diagonalize the inertia tensor. This means that one can always find an orthogonal coordinate system in which the angular momentum and angular velocity are aligned for pure rotation about each of the coordinate axes.
- A rigid body in torque-free motion rotates stably about two of three principal axes, but unstably about the principal axis of intermediate moment of inertia.

8. Proof of Eq. (5)

We wish to prove that

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

First, note that the expression on the left is antisymmetric under the exchanges $j \leftrightarrow k$ and $l \leftrightarrow m$. So is the expression on the right. For the product on the left to be nonzero, each ϵ must be nonzero, which means that ijk must be some permutation of 123. Let us suppose that $j = 2$ and $k = 3$. Then the only nonzero term in ε_{ijk} is when $i = 1$. For the second epsilon symbol to be nonzero, therefore, lm must be either 2 3 or 3 2. In the former case both epsilons are 1 so the product is 1; in the latter, the second is -1, as is the product. Does this work out on the right-hand side?

Let's consider first when $lm = 23$. Then the right-hand side is

$$\delta_{22}\delta_{33} - \delta_{23}\delta_{32} = 1 - 0 = 1$$

as it must be. If $lm = 32$, then the right-hand side becomes

$$\delta_{23}\delta_{32} - \delta_{22}\delta_{33} = 0 - 1 = -1$$

also as it must be.

There is nothing sacred about 2 and 3, of course. Using the permutation properties of the Levi-Civita symbol, we can generate all possible pairs of 2 distinct indices. We have already confirmed that both sides are antisymmetric under these permutations, so the identity is proven.