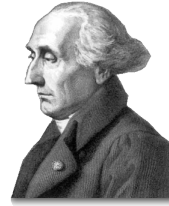


Problem Set 2 Solution

Friday, 21 September 2012

Physics 111



Problem 1 – Tautochrone A particle slides without friction on a cycloidal track given by

$$x = a(\theta - \sin \theta)$$

$$y = a(1 - \cos \theta)$$

where y is oriented vertically downward and x is horizontal. Show that the time taken for the particle to arrive at the bottom of the cycloid ($\theta = \pi$) is $\pi\sqrt{a/g}$, independent of the starting point (which is the meaning of *tautochrone*). (See also problem 3.20 in Helliwell and Sahakian for an interesting literary connection.)

Solution: The time taken to slide down may be computed by integrating $dt = ds/v$. By energy conservation, $E = \frac{1}{2}mv^2 - mgy$ is constant. If we set the zero of energy at the point of release, y_o , then

$$\frac{1}{2}v^2 = g(y - y_o) = ga(\cos \theta_o - \cos \theta) \quad (1)$$

Using the given equations for the cycloid,

$$\dot{x} = a(1 - \cos \theta) \dot{\theta}$$

$$\dot{y} = a \sin \theta \dot{\theta}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 = a^2 \dot{\theta}^2 (1 - 2 \cos \theta + \cos^2 \theta + \sin^2 \theta)$$

$$v^2 = 2a^2 \dot{\theta}^2 (1 - \cos \theta) = 4a^2 \dot{\theta}^2 \sin^2 \frac{\theta}{2}$$

Substituting Eq. (1) into this expression allows us to solve for $\dot{\theta}$,

$$4a^2 \dot{\theta}^2 \sin^2 \frac{\theta}{2} = 2ga(\cos \theta_o - \cos \theta) \rightarrow \dot{\theta} = \sqrt{\frac{g}{2a}} \frac{\sqrt{\cos \theta_o - \cos \theta}}{\sin \frac{\theta}{2}}$$

Separating and integrating, we have

$$\int \sqrt{\frac{g}{2a}} dt = \int \frac{\sin \frac{\theta}{2}}{\sqrt{\cos \theta_o - \cos \theta}} d\theta$$

Using the trig identity, $2 \cos^2 \frac{\theta}{2} = \cos \theta + 1$, we can simplify the integral on the right using the substitution $u = \cos \frac{\theta}{2}$, $du = -\frac{1}{2} \sin \frac{\theta}{2} d\theta$

$$\sqrt{\frac{g}{2a}} t = \int_{\theta_0}^{\pi} \frac{\sin \frac{\theta}{2}}{\sqrt{\cos^2 \frac{\theta_0}{2} - \cos^2 \frac{\theta}{2}}} d\theta = \int_b^0 -\frac{\sqrt{2}}{\sqrt{b^2 - u^2}} du$$

where $b = \cos \theta_0/2$. One more substitution and we're home free: $u = b \sin \phi$, $du = b \cos \phi d\phi$. Then

$$\sqrt{\frac{g}{a}} t = \int_{\pi/2}^0 -\frac{2b \cos \phi}{b \cos \phi} d\phi = \pi$$

Hence, $t = \pi \sqrt{a/g}$.

There has to be a better way. If the time is supposed to be independent of amplitude, then we must be able to express the equation of motion in the form of a simple harmonic oscillator. There are a couple of issues. First, the equations of the cycloid imply that the gravitational potential minimum is at $\theta = \pi$. Since we expect simple harmonic motion about this point, we probably ought to express distances with respect to the minimum. Let $\phi = \theta - \pi$. Then we can rewrite the equations of the cycloid in terms of ϕ to give

$$\begin{aligned} x &= a(\phi + \pi + \sin \phi) \\ y &= a(1 + \cos \phi) \end{aligned}$$

The potential energy with respect to the bottom of the cycloid is

$$U = -mg(y - 2a) = -mga(1 + \cos \phi - 2) = mga(1 - \cos \phi) = 2mga \sin^2 \frac{\phi}{2}$$

The distance s from the potential minimum is given by

$$\begin{aligned} s &= \int_0^{\phi} \sqrt{dx^2 + dy^2} d\phi = \int_0^{\phi} a\sqrt{(1 + \cos \phi)^2 + (-\sin \phi)^2} d\phi \\ &= \int_0^{\phi} a\sqrt{2 + 2\cos \phi} d\phi = \int_0^{\phi} 2a \cos \frac{\phi}{2} d\phi = 4a \sin \frac{\phi}{2} \end{aligned}$$

We may thus express the potential energy directly in terms of the arc length from the bottom as

$$U = \frac{1}{2} \frac{mg}{4a} s^2$$

while the kinetic energy is just

$$T = \frac{1}{2} m \dot{s}^2$$

Hence, the total energy is

$$E = T + U = \frac{1}{2} m \dot{s}^2 + \frac{1}{2} \left(\frac{mg}{4a} \right) s^2$$

which is the equation of a simple harmonic oscillator in s . Hence, the motion is periodic with period independent of the amplitude in s .

Exercise 2 – Transcendental, man! This *Mathematica* exercise relies on the brachistochrone problem, which is discussed in Section 3.4 of Helliwell & Sahakian. For reasons both vague and mysterious, many in NASA are keen on our getting back to the Moon as a stepping stone to putting humans on Mars. This means they will need to construct a launching platform for rockets leaving the Moon for Mars. Suppose it is 200.001-m high, and for undisclosed reasons you are required to construct a track to permit (frictionless) magic moon mushrooms (3M) to be transported solely under gravity from the top of the tower to a point 500.002 m away from the tower's base in minimum time (NASA orders). [Hey, it's at least as plausible as mining the Moon for minerals and it has the virtue that we don't have to worry about air resistance!]

- (a) To 5 significant figures, compute the minimum time for a 3M dropped at rest to make the journey. Take $g = 1.62200 \text{ m/s}^2$ for the value of the gravitational acceleration at the lunar surface.
- (b) How far beneath the lunar surface must the track go at its minimum, to within 1 mm? You may ignore the thickness of the track(!)

Problem Set 2, Exercise 2

The cycloid is given by the equations $x = a(\theta - \sin \theta)$ and $y = a(1 - \cos \theta)$, which passes through the origin. Putting the origin at the top of the tower, the bottom of the ramp must be at $x_1 = 500.002$ m and $y_1 = 200.001$ m, where the y axis is oriented positive down. Dividing these two expressions gets rid of the unknown constant a , allowing us to solve numerically for θ :

$$\begin{aligned} x_1 &= 500.002; \\ y_1 &= 200.001; \\ \theta_{\max} &= \theta /. \text{FindRoot}\left[1 - \text{Cos}[\theta] == \frac{y_1}{x_1} (\theta - \text{Sin}[\theta]), \{\theta, 4\}\right] \\ &3.81966 \end{aligned}$$

Now we can solve for a using the equation for y

$$\begin{aligned} a &= \frac{y_1}{1 - \text{Cos}[\theta_{\max}]} \\ &112.437 \end{aligned}$$

The acceleration due to gravity on the Moon is

$$\begin{aligned} g &= 1.62200 \text{ (* m/s}^2 \text{ *)} \\ &1.622 \end{aligned}$$

To find the time required, we express dt in terms of $d\theta$ via $dt = \frac{ds}{v} = \frac{\sqrt{(x')^2 + (y')^2}}{v} d\theta$, where the primes indicate differentiation with respect to θ . We can express v in terms of θ using the energy conservation condition, $\frac{1}{2} m v^2 = m g a(1 - \cos \theta)$. Hence,

$$dt = \frac{\sqrt{(a(1 - \cos \theta))^2 + (a \sin \theta)^2}}{\sqrt{2 g a (1 - \cos \theta)}} d\theta = \frac{a \sqrt{2 - 2 \cos \theta}}{\sqrt{2 g a (1 - \cos \theta)}} d\theta = \sqrt{a/g} d\theta$$

and

$$\begin{aligned} t &= \sqrt{\frac{a}{g}} \theta_{\max} \\ &31.802 \end{aligned}$$

So, the minimum travel time is 31.802 s. Pretty darn sluggish!

How deep does the trench go? Well, $\theta > \pi$, so the arrival point is after the bottom of the cycloid. The bottom of the cycloid (the greatest value of y) is at $\theta = \pi$, where the value of y is $2a$.

$$\begin{aligned} 2a - y_1 \\ &24.8728 \end{aligned}$$

So, at the deepest, the trench for the track must be 24.873 m deep (on top of the track thickness, which we may neglect).

Problem 3 – A Bowl for cherries A particle of mass m slides without friction inside a spherical bowl of radius R . Using spherical coordinates, write down the Lagrangian. Deduce the equations of motion. Note carefully that you may neglect the size of the particle and you need not *solve* the equations of motion!

Solution: The velocity in spherical coordinates is

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r \dot{\theta} \hat{\boldsymbol{\theta}} + r \sin \theta \dot{\phi} \hat{\boldsymbol{\phi}}$$

so the kinetic energy is

$$T = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2)$$

and the potential energy is $U = mgz = mgR(1 - \cos \theta)$, where I have taken $\theta = 0$ at the bottom of the bowl. Therefore,

$$L = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + mgR(\cos \theta - 1)$$

The ϕ equation of motion is simple:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = 0 \quad \implies \quad mR^2 \sin^2 \theta \dot{\phi} = p^\phi = \text{constant}$$

The equation for θ is more complicated:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0 \quad \implies \quad mR^2 \sin \theta \cos \theta \dot{\phi}^2 - mgR \sin \theta - \frac{d}{dt} (mR^2 \dot{\theta}) = 0$$

Dividing through by $-mR^2$ we get

$$\ddot{\theta} + \frac{g}{R} \sin \theta - \sin \theta \cos \theta \dot{\phi}^2 = 0$$

$$\ddot{\theta} + \frac{g}{R} \sin \theta - \frac{(p^\phi)^2}{m^2 R^4} \cot \theta \csc^2 \theta = 0$$

Problem 4 – Loop the loop A sphere of mass m and radius a rolls without slipping inside a semicircular track of radius R that occupies a vertical plane. Let ϕ represent the angle of rotation of the sphere, and let θ measure the angle between the center of the semicircle and the center of the sphere, with respect to the vertical. That is, at $\theta = 0$, the sphere is at the bottom of the track.

- Carefully deduce the equation of constraint relating θ and ϕ . Use a large, clear diagram to clarify the argument. Hint: the equation is *not* $R\Delta\theta = -a\Delta\phi$.
- Noting that the rotational kinetic energy of the sphere may be expressed

$$kma^2 \dot{\phi}^2 / 2$$

where the constant k depends on the radial dependence of the mass density (and is $2/5$ for a uniform distribution), and that the total kinetic energy is the sum of the rotational kinetic energy and the translation kinetic energy of the sphere's center of mass, write down the Lagrangian.

- Use the constraint equation to eliminate ϕ from L .
- Derive the equation of motion.

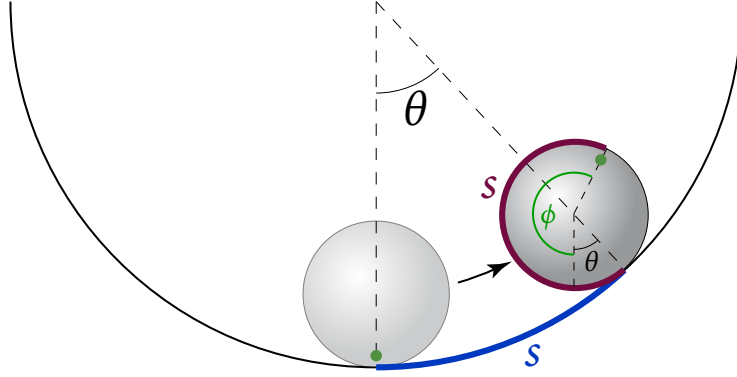


Figure 1: As the sphere of radius a rolls along the surface of the track, the arc length along the track is equal to the arc length along the surface of the sphere.

- (e) Calculate the period of small oscillations about the bottom of the track by considering small displacements from equilibrium.

Solution:

(a) Rolling without slipping means that the distance along the bowl, $s = R\theta$ is equal to the distance along the surface of the sphere, which is $s = a(-\phi + \theta)$, as illustrated in Fig. 1. Assuming that both θ and ϕ use the same counterclockwise direction as positive, as θ increases, ϕ decreases, which is why I have inserted the minus sign. Equating the two expressions for s , we get $-\phi = (R/a - 1)\theta$.

(b) The center of the sphere travels on a circle of radius $R - a$, so the translational kinetic energy is $\frac{1}{2}m(R - a)^2\dot{\theta}^2$. The rotational kinetic energy is $\frac{1}{2}kma^2\dot{\phi}^2$, and the gravitational potential energy is $mg(R - a)(1 - \cos \theta)$. Putting these together we have

$$L = \frac{1}{2}m(R - a)^2\dot{\theta}^2 + \frac{1}{2}kma^2\dot{\phi}^2 - mg(R - a)(1 - \cos \theta)$$

(c) Using the constraint equation to eliminate $\dot{\phi}$, we get

$$\begin{aligned} L &= \frac{1}{2}m(R - a)^2\dot{\theta}^2 + \frac{1}{2}kma^2\left(\frac{R - a}{a}\right)^2\dot{\theta}^2 - mg(R - a)(1 - \cos \theta) \\ &= \frac{1}{2}m(1 + k)(R - a)^2\dot{\theta}^2 - mg(R - a)(1 - \cos \theta) \end{aligned}$$

(d)

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &\implies -mg(R - a)\sin \theta - \frac{d}{dt}\left(m(1 + k)(R - a)^2\dot{\theta}\right) = 0 \\ \ddot{\theta} + \frac{g}{(1 + k)(R - a)}\sin \theta &= 0 \end{aligned}$$

For small oscillations, we may approximate $\sin \theta \approx \theta$, giving the simple harmonic oscillator equation,

$$\ddot{\theta} + \omega^2 \theta = 0$$

where $\omega = \sqrt{g/[(1+k)(R-a)]}$. Therefore, the period of small oscillations is

$$P = 2\pi\sqrt{(1+k)(R-a)/g}$$