BEYOND EQUATIONS

James Clerk Maxwell (1831 - 1879)

J. C. Maxwell was born in Edinburgh, Scotland. He was educated at home by his mother, who died when Maxwell was eight, and then by his father. He attended the University of Edinburgh starting at age sixteen, and then Cambridge University, starting at age nineteen. He obtained a professorship at Aberdeen University, where he taught for four years before he was laid off due to a merger of two institutions. He then became a professor at King's College London, where he spent five years: this was the most productive period of his life.

Building upon physical concepts of Michael Faraday and others, Maxwell formulated the mathematical theory of electromagnetism, uniting electrical and magnetic phenomena, and showing that light is an electromagnetic wave. This grand unification was the most important advance in physics during the nineteenth century. Maxwell also contributed to the development of Maxwell Boltzmann statistics in statistical mechanics; and worked on the theory of color, the viscosity of gases, and dimensional analysis, He wrote a textbook, *Theory of Heat*, and an elementary monograph, *Matter and Motion*.

Maxwell resigned his chair at King's College in 1865 and returned to his home to Edinburgh. In 1871 he was named the first Cavendish Professor at Cambridge; he remained there for seven years, building up the Cavendish Laboratory until his death of cancer at the age of 48, the same form of abdominal cancer that had killed his mother. As of 2011, 29 Cavendish researchers have won Nobel Prizes.

In the view of many, Maxwell is the third greatest physicist of all time. Biographies of the first two can be found in the first two chapters of this book.

Chapter 8

Electromagnetism

While gravity was the first of the fundamental forces to be quantified and at least partially understood – all the way back in the 17th century – it took an additional 200 years for physicists to unravel the secrets of a second fundamental force, the electromagnetic force. Ironically, it is the electromagnetic force that is by far the stronger of the two, and at least as prevalent in our daily lives. The fact that atoms and molecules stick together to form the matter we are made of, the contact forces we feel when you touch any object around us, and virtually all modern technological advances of the twentieth century, all these rely on the electromagnetic force. In this chapter, we introduce the subject within the Lagrangian formalism and demonstrate some familiar as well as unfamiliar aspects of this fascinating fundamental force law of Nature.

8.1 The Lorentz force law

In the mid-nineteenth century, the Scottish physicist James C. Maxwell (1831-1879) combined the results of many experimental observations having to do with magnets and fuzzy cats, and formulated a set of equations describing a new force law, the electromagnetic force. Maxwell's equations in vacuum can be written in Gaussian units as

$$\begin{aligned} \nabla \cdot \boldsymbol{E} &= 4\pi\rho \quad & \nabla \times \boldsymbol{E} &= -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \\ \nabla \cdot \boldsymbol{B} &= 0 \quad & \nabla \times \boldsymbol{B} &= \frac{4\pi}{c}\boldsymbol{J} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \ . \end{aligned} \tag{8.1}$$

Here $\boldsymbol{E}$ and $\boldsymbol{B}$ are the space-time dependent electric and magnetic vector fields. Through perfect vacuum, they can relay a force between objects that carry an attribute called *electric charge*. The quantities ρ and $\boldsymbol{J}$ are the charge density and current density of the charged stuff that is causing the corresponding $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. For example, a single, isolated, and stationary point particle of electric charge q located at a position $\boldsymbol{r}_0$ would be described by

$$\rho = q\,\delta^3(\boldsymbol{r}-\boldsymbol{r}_0) \quad , \quad \boldsymbol{J}=0\ , \tag{8.2}$$

where $\delta^3(\boldsymbol{r}-\boldsymbol{r}_0)$ is the three-dimensional **delta function**, which is infinite at $\boldsymbol{r}=\mathbf{r}_0$, zero elsewhere, and whose volume integral is unity as long as the location $\boldsymbol{r}_0$ is contained within the volume in question. The corresponding electric and magnetic fields are obtained from Maxwell's equations (8.1) as (with $\boldsymbol{R}\equiv\boldsymbol{r}-\boldsymbol{r}_0$)

$$\boldsymbol{E}=\frac{q}{|\boldsymbol{R}|^2}\hat{\boldsymbol{R}} \quad , \quad \boldsymbol{B}=0 \tag{8.3}$$

known as the **Coulomb field** profile. Another classical example is that of a charge q moving with constant velocity $\boldsymbol{v}$. At a position $\boldsymbol{R}$ away from the charge, one gets the **Biot-Savard** magnetic field profile

$$\boldsymbol{B}=\frac{q\boldsymbol{v}\times\boldsymbol{R}}{c\,R^3}\ . \tag{8.4}$$

Given a probe particle of mass M and electric charge Q in the presence of electric and magnetic fields – generated by other nearby charges described by ρ and $\boldsymbol{J}$ – the force that the probe particle feels is given by

$$\boldsymbol{F}_{\rm em}=Q\boldsymbol{E}+\frac{Q}{c}\boldsymbol{v}\times\boldsymbol{B}\ , \tag{8.5}$$

known as the **Lorentz force**. For example, taking the environment of the probe as consisting of the point charge q of (8.2)) and (8.3, the probe feels a force given by

$$\boldsymbol{F}=\frac{Q\,q}{|\boldsymbol{r}-\boldsymbol{r}_0|^2}\hat{\boldsymbol{r}} \tag{8.6}$$

with the probe charge Q located at $\boldsymbol{r}$ and the source charge q located at $\boldsymbol{r}_0$ (see Figure 8.1).

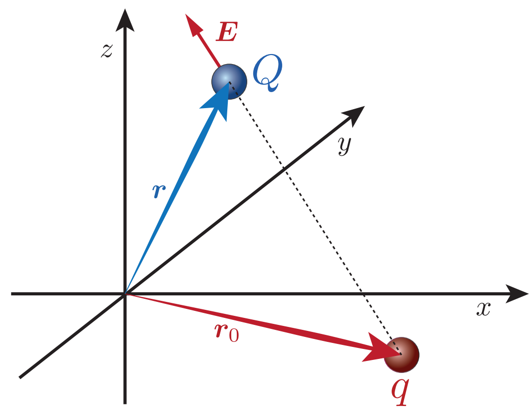


FIG 8.1 : The electrostatic Coulomb force between two charged particles.

This force law should look very familiar. Remember that the *gravitational* force experienced by a probe mass M located at $\boldsymbol{r}$ in the vicinity of a source mass m at $\boldsymbol{r}_0$ is given by

$$\boldsymbol{F} = -G\frac{M\,m}{|\boldsymbol{r}-\boldsymbol{r}_0|^2}\hat{\boldsymbol{r}}\ . \tag{8.7}$$

Instead of the product $-G\,M\,m$ in (8.7)), the strength of the electromagnetic force is tuned by the product of the charges $q\,Q$ as in (8.6. The rest, the inverse square distance law, is the same. This is not a coincidence. Both forces have a geometrical origin and are tightly constrained by similar symmetries of Nature. In the capstone chapter of this section, we will explore these similarities further.

Equations (8.1)) consist of eight differential equations for the six fields tucked within $\boldsymbol{E}$ and $\boldsymbol{B}$ – sourced by some charge distribution described by $\boldsymbol{\rho}$ and $\boldsymbol{J}$. An existence and uniqueness theorem of the theory of differential equations guarantees that, given $\boldsymbol{\rho}$ and $\boldsymbol{J}$, equations (8.1) always determine $\boldsymbol{E}$ and $\boldsymbol{B}$ uniquely. In turn, each of the charges making up ρ and $\boldsymbol{J}$ experiences the Lorentz force (8.5) and evolves accordingly; which in turn changes the electric and magnetic fields via (8.1. Hence, we have a coupled set of differential equations for $\boldsymbol{E}$, $\boldsymbol{B}$, and the position of the charges – equations that are to be solved in principle simultaneously.

In practice, this is a very hard problem. Fortunately, in many practical circumstances there is a clear separation of roles between the charges involved in the electromagnetic interactions. Some of the charges – called the *source* charges – have fixed and given dynamics and can be used to compute the electric and magnetic fields in a region of interest: that is, given ρ and $\boldsymbol{J}$, one uses (8.1)) to find $\boldsymbol{E}$ and $\boldsymbol{B}$. The remaining charges of interest are called *probe* charges. The electromagnetic fields they generate are negligible compared to the ones from the source charges, and their dynamics is described by the Lorentz force law (8.5 with given $\boldsymbol{E}$ and $\boldsymbol{B}$ background fields. This approximation scheme decouples the set of differential equations into two separate, more tractable, sets. In this text, we focus on the second problem: the mechanics of probe charges in the background of *given* electric and magnetic fields.

The task at hand is then the following. Given some $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, we want to study the dynamics of point charges using the Lagrangian formalism. Hence, we want to incorporate the Lorentz force law (8.5)) into a Lagrangian – and hence a variational principle. We start by rewriting the background electric and magnetic fields in terms of new fields that make the underlying symmetries of Maxwell's equations more apparent. From (8.1, we know that the magnetic field is divergenceless. This implies that we can write

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \tag{8.8}$$

trading the $\boldsymbol{B}$ field for a new vector field $\boldsymbol{A}$ we call the **vector potential**. Once again from (8.1), we have

$$\nabla \times \left(\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}\right) = 0 \tag{8.9}$$

which implies that we can write

$$\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} = -\nabla\phi \Rightarrow \boldsymbol{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \tag{8.10}$$

introducing a new *scalar* field ϕ which we call the **scalar potential**. Hence, we have traded the six fields in $\boldsymbol{E}$ and $\boldsymbol{B}$ for four fields $\boldsymbol{A}$ and ϕ. The fact that we can do so is a reflection of a deep and foundational symmetry underlying the electromagnetic force law. Given $\boldsymbol{E}$ and $\boldsymbol{B}$, even $\boldsymbol{A}$ and ϕ are *not* unique. We can apply the following transformations to $\boldsymbol{A}$ and ϕ without changing $\boldsymbol{E}$ and $\boldsymbol{B}$ – and hence the force law

$$\boldsymbol{A} \to \boldsymbol{A} + \boldsymbol{\nabla}\Phi \quad , \quad \phi \to \phi - \frac{1}{c}\frac{\partial \Phi}{\partial t} \tag{8.11}$$

with any arbitrary function of spacetime Φ. Thus, there is even less physical information in $\boldsymbol{A}$ and ϕ. This remarkable symmetry of the theory is called **gauge symmetry**. Indeed, it is possible to *derive* electromagnetism based solely on the principles of relativity and gauge symmetry.

EXAMPLE 8-1: Fixing a gauge

The gauge symmetry (8.11) provides for a freedom in choosing a particular scalar and vector potential for a given electric and magnetic fields. That is, one may have many profiles for the potentials correspond to one physical electromagnetic field profile. Given this freedom, it is customary to **fix the gauge** so as to make the manipulation of the potentials more convenient. For example, we may choose the **static gauge**

$$\phi = 0 \quad \text{Static gauge} \tag{8.12}$$

We can see that this is always possible as follows. Imagine you start with some ϕ and $\boldsymbol{A}$ such that $\phi \neq 0$. Then apply a gauge transformation (8.11) – which we know does not change the electromagnetic fields and the associated physics – such that

$$\phi' = \phi - \frac{1}{c}\frac{\partial \Phi}{\partial t} = 0 \; . \tag{8.13}$$

That is, find a function Φ such that this equation is satisfied. For any ϕ, this equation indeed has a solution Φ. This is a rather strange *gauge choice* since it sets the electric potential to zero. But this is entirely legal. Note that it does *not* imply that the electric field is zero since we still have

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \tag{8.14}$$

in this gauge choice. Another interesting aspect of gauge fixing is that, typically, the process does not necessarily fix *all* the gauge freedom. In the case of the static gauge, we can still apply a gauge transformation Φ_0 such that

$$\phi' = 0 \to \phi'' = 0 = \phi' - \frac{1}{c}\frac{\partial \Phi_0}{\partial t} = -\frac{1}{c}\frac{\partial \Phi_0}{\partial t} \tag{8.15}$$

without change the gauge condition that the electric potential is zero. We see from this expression that this is possible if

$$\frac{\partial \Phi_0}{\partial t} = 0 \; ; \tag{8.16}$$

that is, if the gauge transformation function Φ_0 is time independent. Hence, some of the original freedom of the gauge symmetry is still left even after gauge fixing. This is known as **residual gauge freedom** for obvious reasons.

Another very common gauge choice is the **Coulomb gauge**

$$\boldsymbol{\nabla}\cdot\boldsymbol{A}=0\ ; \tag{8.17}$$

and a Lorentz invariant version known as the **Lorentz gauge**

$$\frac{\partial A_\mu}{\partial x_\nu}\eta_{\mu\nu}=0\ . \tag{8.18}$$

In the problem section, you will show that both of these conditions are possible; and in fact they are each associated with residual gauge freedoms.

8.2 The Lagrangian for electromagnetism

Maxwell's equations – inspired by a series of experiments – are supposed to describe a law of physics in an inertial reference frame. Given that they imply that light propagates in vacuum with a universal speed c, Galilean transformations cannot be valid, and Lorentz transformations are established to link the perspectives of inertial observers. This is how relativity was historically developed. The question we now want to address is the following: if an inertial observer $\mathcal{O}$ measures an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$, what would the electric field $\boldsymbol{E}'$ and magnetic field $\boldsymbol{B}'$ be as measured by an observer $\mathcal{O}'$ – moving as usual with constant velocity $\boldsymbol{v}$ along the common x or x' axis. We assume that the transformation relating these fields is linear in the fields, much like the Lorentz transformation of four-position or four-velocity; we also assume that they are linear in the relative velocity $\boldsymbol{v}$. We start with expressions of the form

$$\boldsymbol{E}'=\hat{\boldsymbol{A}}_1\cdot\boldsymbol{E}+\hat{\boldsymbol{A}}_2\cdot\boldsymbol{B}\quad,\quad\boldsymbol{B}'=\hat{\boldsymbol{A}}_3\cdot\boldsymbol{E}+\hat{\boldsymbol{A}}_4\cdot\boldsymbol{B} \tag{8.19}$$

with the $\hat{\boldsymbol{A}}_i$ being four 3 X 3 matrices whose components can depend on $\boldsymbol{v}$ at most linearly. And we require that the Lorentz force law (8.5) fits as the last three components of a four-force, as seen from Chapter 2. With all these conditions in place, one finds a unique solution for the $\hat{\boldsymbol{A}}_i$'s. One can show that, given $\boldsymbol{E}$ and $\boldsymbol{B}$ as measured by an inertial observer $\mathcal{O}$, another inertial observer $\mathcal{O}'$ moving with respect to $\mathcal{O}$ with velocity $\boldsymbol{v}$ measures different electric and magnetic fields $\boldsymbol{E'}$ and $\boldsymbol{B'}$ given by

$$\boldsymbol{E'}=\gamma\left(\boldsymbol{E}+\frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right)+(1-\gamma)\frac{\boldsymbol{E}\cdot\boldsymbol{v}}{v^2}\boldsymbol{v} \tag{8.20}$$

$$\boldsymbol{B'} = \gamma\left(\boldsymbol{B} - \frac{\boldsymbol{v}}{c} \times \boldsymbol{E}\right) + (1-\gamma)\frac{\boldsymbol{B}\cdot\boldsymbol{v}}{v^2}\boldsymbol{v} \tag{8.21}$$

These rather complicated relations become more transparent when written in terms of ϕ, $\boldsymbol{A}$ and ϕ', $\boldsymbol{A'}$. Introduce a four vector

$$A_\mu = (\phi, \boldsymbol{A}) \tag{8.22}$$

one can show that we simply have

$$A_{\mu'} = \Lambda_{\mu'\mu} A_\mu \tag{8.23}$$

where $\Lambda_{\mu'\mu}$ is the usual Lorentz transformation matrix of Chapter 2. Hence, the information about the electromagnetic fields is now packaged in a four-vector A_μ that transforms in a simple manner under Lorentz transformations.

All this allows us to develop a variational principle for the electromagnetic force law using Lorentz symmetry. We start with the familiar relativistic action

$$S = -m\,c^2 \int d\tau = -m\,c^2 \int dt\,\sqrt{1-\frac{v^2}{c^2}} \tag{8.24}$$

for a free point mass m. And we want to add a term to this action such that the particle experiences the Lorentz force law as if it had a charge Q in some background ϕ and $\boldsymbol{A}$ fields; and this combined action better be Lorentz invariant[1]. Looking back at the Lorentz force law (8.5), noting in particular that it is linear in the particle velocity and the background fields, we can write a unique Lorentz-invariant integral consistent with these statements and Lorentz symmetry:

$$\int A_\mu \eta_{\mu\nu} \frac{dx^\nu}{d\tau} d\tau\ . \tag{8.25}$$

Adding an appropriate multiplicative constant, we get the full action

$$S = -m\,c^2 \int d\tau + \frac{Q}{c}\int A_\mu \eta_{\mu\nu} dx^\nu . \tag{8.26}$$

[1]In fact, as mentioned earlier in the book, the principle of invariance of this action under Lorentz transformations was the guiding principle for developing relativity and the associated Lorentz transformations. It may appear as a chicken and egg problem; in reality, one should think of the Lorentz symmetry as the fundamental requirement, and the physical consequences as Relativity and Electromagnetism.

We then expand the second term in more detail to get

$$\begin{aligned}\frac{Q}{c}\int A_\mu \eta_{\mu\nu} dx^\nu &= \frac{Q}{c}\int A_\mu \eta_{\mu\nu}\frac{dx^\nu}{dt}dt = \frac{Q}{c}\int\left(-\phi\, c\, dt + \boldsymbol{A}\cdot\frac{d\boldsymbol{r}}{dt}dt\right) \\ &= Q\int dt\left(-\phi + \boldsymbol{A}\cdot\frac{\boldsymbol{v}}{c}\right)\ . \end{aligned} \tag{8.27}$$

We leave it as an exercise for the reader to now check that the resulting equations of motion from (8.26) reproduce the Lorentz force law

$$\frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}\left(\gamma_v m\, \boldsymbol{v}\right) = Q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c}\times\boldsymbol{B}\right) = \boldsymbol{F}_{\mathrm{em}} \tag{8.28}$$

We thus have a Lagrangian formulation of the electromagnetic force law.

In the non-relativistic limit, the action (8.26) becomes

$$L = \frac{1}{2}m\dot{x}_i^2 - q\phi + \frac{q}{c}A_i\dot{x}_i \tag{8.29}$$

with equations of motion

$$m\boldsymbol{a} = q\boldsymbol{E} + \frac{q}{c}\boldsymbol{v}\times\boldsymbol{B}\ , \tag{8.30}$$

dropping terms quadratic in v/c while keeping linear terms. This set of equations will be our focus in the next several examples.

8.3 The two-body problem, once again

We start in a familiar place: the two-body problem, now with electromagnetic rather than gravitational interactions. Two point particles of masses m_1 and m_2, located at $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ respectively, carry electric charges q_1 and q_2. Let us first roughly estimate the relative importance of the various forces involved. Electromagnetic fields propagate with the speed of light in vacuum. This implies that if the particles are moving slowly compared to the speed of light, we may think of the electromagnetic fields cast about them as propagating virtually instantaneously – always reflecting their instantaneous positions. The electric field $\boldsymbol{E}$ from the point charge q_2 a distance $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ away generates a force on q_1 of the order $F_{\mathrm{el}} = q_1 E \sim q_1 q_2/r^2$. If q_2 is moving with a speed $v_2 \ll c$ while q_1 is moving with $v_1 \ll c$, q_1 experiences a magnetic force of the order of $F_{\mathrm{m}} \sim q_1 v_1 B/c \sim q_1 q_2 v_1 v_2/c^2\, r^2$. Let us also mention that accelerating charges radiate electromagnetic energy, which can add a level of complication. However, once again, this effect is much smaller in the non-relativistic regime. Finally, the two masses interact gravitationally with a force of the order of $F_g \sim G m_1 m_2 / r^2$. Putting things together, we summarize

$$F_{\rm el} \ : \ F_m \ : \ F_g \ \sim \frac{q_1 q_2}{r^2} \ : \ \frac{q_1 q_2 v_1 v_2}{c^2 r^2} \ : \ \frac{G m_1 m_2}{r^2} \ . \tag{8.31}$$

Since $v_1 v_2 \ll c^2$, we see that the magnetic force between the charges is less than the electric force by a factor v^2/c^2. To compare the electric force to the gravitational one, consider the case of two electrons with mass $m \simeq 9 \times 10^{-28}$ g and charge $q \simeq 5 \times 10^{-10}$ esu. We get an estimate for $F_e \ : \ F_g \sim 1 \ : \ 10^{-43}$... Phrasing things gently, we need not care about gravitational forces! Gravity becomes relevant only when we are dealing with electrically neutral matter with $q = 0$, like entire atoms and planets.

Thus, we need to care only about the electrostatic force acting between the two point charges. We write the Lagrangian

$$L = \frac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \frac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - U(r) \tag{8.32}$$

where $U(r) = q_1 \phi(r)$, with $\phi(r)$ being the scalar potential due to source charge q_2 at the location of the probe charge q_1

$$\phi = \frac{q_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} = \frac{q_2}{r} \ . \tag{8.33}$$

This gives the electric potential energy

$$U(r) = \frac{q_1 q_2}{r} \ . \tag{8.34}$$

We now have a familiar two-body problem with a central potential! We can then import the entire machinary developed in chapter 7. We first factor away the trivial center of mass motion and write a Lagrangian for the interesting relative motion tracked by $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$,

$$L = \frac{1}{2} \mu \dot{\boldsymbol{r}}^2 - \frac{q_1 q_2}{r} \tag{8.35}$$

where $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. This quickly leads to a one-dimensional problem in the radial direction with fixed energy

$$E = \frac{1}{2} \mu \dot{r}^2 + U_{\rm eff}(r) \tag{8.36}$$

and effective potential

$$U_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + \frac{q_1 q_2}{r} \tag{8.37}$$

where l is the conserved angular momentum. All is very similar to the problem of two point masses interacting gravitationally, except for the following important observations:

- If $q_1 q_2 < 0$, *i.e.*, the charges are of opposite signs, the electric force between the point particles is *attractive*. Our entire analysis of orbits and trajectories from the gravitational analogue goes through with the simple substitution

  $$-G m_1 m_2 \to q_1 q_2 \tag{8.38}$$

  in all equations. We will then find closed orbits consisting of circles and ellipses, and open orbits consisting of parabolas and hyperbolas. For example, the radius of a stable circular orbit would be given by

  $$r_p = \frac{|G\, m_1\, m_2|}{2|E|} \to \frac{|q_1 q_2|}{2|E|} \ . \tag{8.39}$$

  For a hydrogen atom with energy $|E| \simeq 13.6$ eV, we would get $r_p \sim 10^{-8}$ cm, a good estimate for the size of the ground state atom.

- If $q_1 q_2 > 0$, the electric force is *repulsive*, a situation that does not happen for gravitation. Let us then look at this case more closely.

When $q_1 q_2 > 0$, the formalism developed in Chapter 7 still goes through, except we need to be careful with certain signs. For example, the orbit trajectory becomes

$$r = \frac{(l^2/G\, m_1 m_2)\,(1/m_1)}{1 + e\cos\theta} \to \frac{(l^2/q_1 q_2 m_1)}{1 + e\cos\theta} \tag{8.40}$$

with the eccentricity

$$\epsilon = \sqrt{1 + \frac{2El^2}{(G^2 m_1^2 m_2^2)\, m_1}} \to \sqrt{1 + \frac{2El^2}{(q_1^2 q_2^2)\, m_1}} \ . \tag{8.41}$$

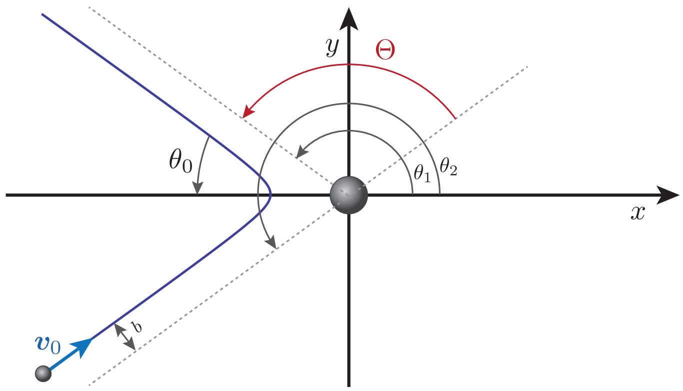


FIG 8.2 : Hyperbolic trajectory of a probe scattering off a charged target.

However, since $q_1 q_2 > 0$, we now see that we necessarily have $E > 0$, and hence

$$\epsilon > 1 \ . \tag{8.42}$$

This implies that the trajectory is necessarily hyperbolic. Obviously, with a repulsive force, we may not have bound orbits. The interesting physics problem is that of particle scattering.

8.4 Coulomb scattering

Consider a point charge q_1 projected with some initial energy from infinity onto point charge q_2 as shown in Figure 8.2. We say the repulsive electrostatic force from q_2 scatters the probe at an angle that can be read off from (8.40) as

$$r \to \infty \Rightarrow \cos\theta_{1,2} \to -\frac{1}{\epsilon} \tag{8.43}$$

as shown in the figure. The scattering angle Θ is defined as

$$\theta_1 - \theta_2 = 2\theta_0 = \pi - \Theta \ . \tag{8.44}$$

We then easily get

$$\cos\Theta = \frac{2El^2/m_1 - q_1^2 q_2^2}{2El^2/m_1 + q_1^2 q_2^2} \ . \tag{8.45}$$

It is convenient to write the angular momentum l in terms of the so-called **impact parameter** b shown in the figure. Looking at the initial configuration at $r \to \infty$, we have the angular momentum l given by

$$l = m_1 v_0 b = b\sqrt{2\, m_1 E} \tag{8.46}$$

where v_0 is the initial speed of the probe, related to the constant energy E. We then get

$$\cos\Theta = \frac{4\, b^2 E^2 - q_1^2 q_2^2}{4\, b^2 E^2 + q_1^2 q_2^2} \ . \tag{8.47}$$

Using the trigonometric identity

$$\cot^2 \frac{\Theta}{2} = \frac{(1+\cos\Theta)^2}{(1-\cos^2\Theta)} \ , \tag{8.48}$$

we can simplify this expression further to

$$\cot \frac{\Theta}{2} = \frac{2\, bE}{q_1 q_2} \ . \tag{8.49}$$

This relation tells us the scattering deflection angle a point charge q_1 experiences, when projected from infinity with energy E and impact parameter b onto another point charge q_2. A scattering process such as this is a powerful experimental probe into atomic structure and was instrumental in discovering the constituents of atoms. Put simply, one uses the electromagnetic force to poke into the electrically charge universe of the atom – by simply throwing charged particles at it.

While the initial energy E of the probe can be controlled, the impact parameter b is in practice impossible to measure on a per scattering atom basis. It is then useful to describe scattering processes through a quantity called the **scattering cross-section**: we look at the change in the area within which a probe scatters away in relation to a change in the impact parameter, as shown in Figure 8.3. The scattering cross section $\sigma(\Theta)$ is defined as the change in initial impact area per change in scattering area on a unit sphere centered at the target

$$\sigma(\Theta) \equiv \left|\frac{2\pi b db}{2\pi \sin\Theta d\Theta}\right| = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| \ . \tag{8.50}$$

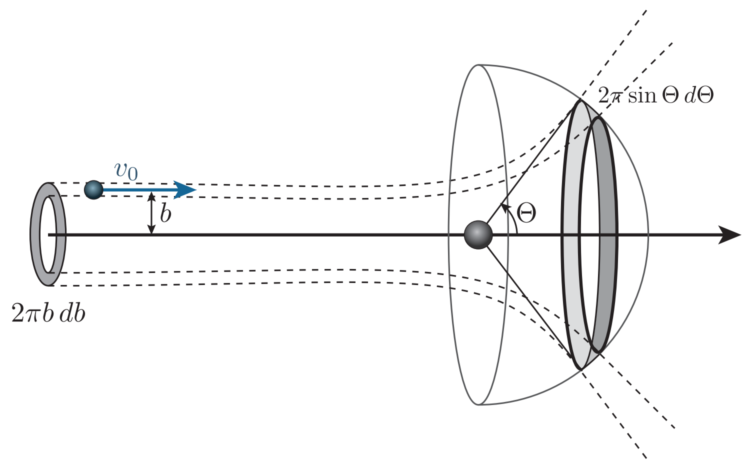


FIG 8.3 : Definition of the scattering cross section in terms of change in impact area $2\pi b db$ and scattering area $2\pi \sin\Theta d\Theta$ on the unit sphere centered at the target.

If you think of a stream of incident probe particles falling onto the target at various unknown impact parameters b with a uniform distribution in b, $\sigma(\Theta)$ is then proportional to the probability of finding a scattered probe at an angle Θ on the unit sphere centered at the target. In the celebrated case of Coulomb scattering described in this example, one gets the so-called **Rutherford scattering** cross section.

$$\sigma(\Theta) = \frac{1}{4}\left(\frac{q_1^2 q_2^2}{2\,E}\right)^2 \csc^4\frac{\Theta}{2} \tag{8.51}$$

As shown in Figure 8.4, this probability is sharply peaked in the forward $\Theta = 0$ direction. It also has a strong dependence on the charges through a fourth power. $\sigma(\Theta)$ can be readily measured, and given E for example, used to determine the charge of the target. A scattering process is effectively a way to looking into atoms using charges – as we look into say neutral biological tissue using scattered light and a microscope.

EXAMPLE 8-2: Snell scattering

As an illustration of the concept of scattering cross section, consider the scattering of light off a perfectly polished bead whose surface acts like a mirror. The bead's radius is R as shown

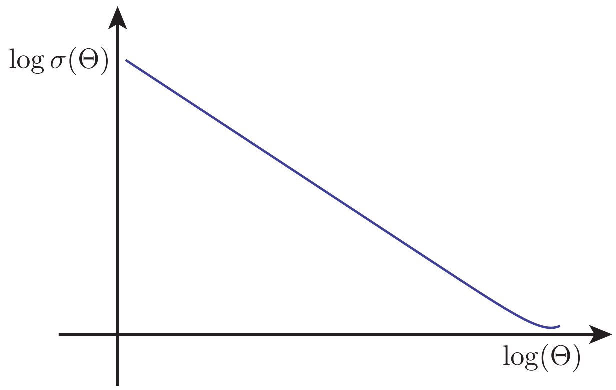


FIG 8.4 : The Rutherford scattering cross section. The graph shows $\log \sigma(\Theta)$ as a function of $\log \Theta$ superimposed on actual data in scattering of protons off gold atoms.

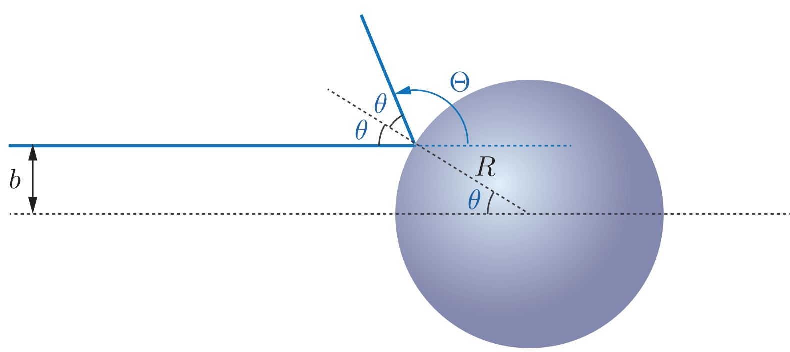


FIG 8.5 : Scattering of light off a reflecting bead.

in Figure 8.5. We want to find the scattering cross section of this bead as parallel light falls on it.

We start from equation (8.50). We then need to find $b(\Theta)$ using Snell's law. Looking at the figure, we can read off

$$b(\Theta) = R\,\sin\theta = R\,\sin\left(\pi - \frac{\Theta}{2}\right) = R\,\sin\left(\frac{\Theta}{2}\right) \ . \tag{8.52}$$

Using this in (8.50), we get

$$\sigma(\Theta) = \frac{1}{4} b\,R\,\csc\frac{\Theta}{2} = \frac{1}{4}\frac{b\,R}{\sin\theta} = \frac{1}{4}R^2 = \frac{1}{4\pi}\left(\pi R^2\right) \ . \tag{8.53}$$

That is the *total* scattering cross section is

$$\sigma_T = \int_0^{2\pi}\int_0^{\pi} \sigma(\Theta)\sin\Theta d\Theta d\Phi = 4\pi\frac{1}{4\pi}\left(\pi R^2\right) = \pi R^2 \ , \tag{8.54}$$

which is simply the cross sectional area of the bead! As the interaction of the in-falling probe with the target becomes longer ranged, the total scattering cross section increases: it is as if the cross section size seen by probes expands because of the longer range of the interactions.

8.5 Uniform magnetic field

Consider a point particle of mass m moving in the background of some given *static* magnetic field with no electric fields. The Lagrangian is then given by

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{q}{c}A_i\dot{x}_i \ . \tag{8.55}$$

If the background magnetic field $\boldsymbol{B}$ is *uniform*, as in

$$\boldsymbol{B} = B\hat{\boldsymbol{z}} \ , \tag{8.56}$$

we can write a vector potential corresponding to this magnetic field as

$$\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{r} \ . \tag{8.57}$$

Note that due to the gauge symmetry (8.11), this choice of $\boldsymbol{A}$ is not unique. but is convenient to use in this case. We then have

$$\boldsymbol{A} = -\frac{1}{2}By\hat{\boldsymbol{x}} + \frac{1}{2}Bx\hat{\boldsymbol{y}} \ . \tag{8.58}$$

The Lagrangian becomes

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{q\,B}{2\,c}\dot{x}y + \frac{q\,B}{2\,c}\dot{y}x \ , \tag{8.59}$$

with corresponding equations of motion

$$m\ddot{x} = \frac{q\,B}{c}\dot{y} \quad , \quad m\ddot{y} = -\frac{q\,B}{c}\dot{x} \quad , \quad m\ddot{z} = 0 \ . \tag{8.60}$$

The dynamics in the z direction decouples and is rather trivial. Setting initial conditions

$$z(0) = 0 \quad , \quad \dot{z}(0) = V_z \tag{8.61}$$

we get

$$z(t) = V_z t; \tag{8.62}$$

hence, the particle moves with uniform speed along the z axis as a free particle. In the x and y directions however, we have a more interesting scenario. We can integrate the $\ddot{x}$ and $\ddot{y}$ equations immediately, and get

$$\dot{x} = \omega_0\left(y - y_0\right) \quad , \quad \dot{y} = -\omega_0\left(x - x_0\right) \tag{8.63}$$

where

$$\omega_0 \equiv \frac{q\,B}{m\,c} \ , \tag{8.64}$$

which is called the *cyclotron frequency.* Here x_0 and y_0 are constants of integration whose role will become apparent shortly. The new equations (8.63) suggest the change of variable

$$X \equiv x - x_0 \quad , \quad Y \equiv y - y_0 \ , \tag{8.65}$$

to yield a somewhat simpler set of coupled equations

$$\dot{X} = \omega_0 Y \quad , \quad \dot{Y} = -\omega_0 X \ . \tag{8.66}$$

These can be decoupled quickly by differentiating with respect to time

$$\ddot{X} = -\omega_0^2 X \quad , \quad \ddot{Y} = -\omega_0^2 Y \tag{8.67}$$

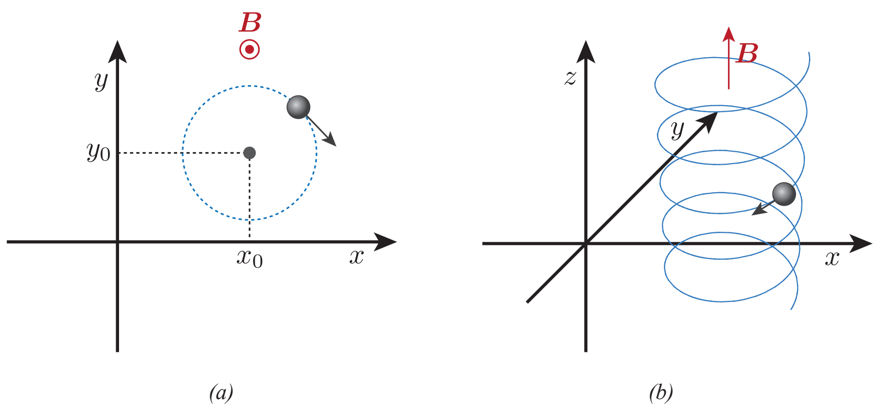


FIG 8.6 : (a) Top view of a charged particle in a uniform magnetic field; (b) The helical trajectory of the charged particle.

leading us to familiar harmonic oscillators. We now see that the point charge would be circling in the x, y plane, about the point (x_0, y_0), as shown in Figure 8.6(a). If $X > 0$, we have $\dot{Y} < 0$ with $\omega_0 > 0$. This implies that if $q\,B > 0$, the circling is in the clockwise direction in the $x - y$ plane, as seen looking down the z axis.

Let us choose a set of convenient initial conditions: first,

$$Y(0) = 0 \Rightarrow \dot{X}(0) = 0 \tag{8.68}$$

since $\dot{X} = \omega_0 Y$. Next,

$$X(0) = R \Rightarrow \dot{Y}(0) = V_y = -\omega_0 R \tag{8.69}$$

since $\dot{Y} = -\omega_0 X$, where we have denoted the radius of the circular trajectory as R. We then get trajectory

$$X(t) = R\cos(\omega_0 t) \quad , \quad Y(t) = -R\sin(\omega_0 t) \ . \tag{8.70}$$

In the original coordinates, it is given by

$$x(t) - x_0 = R\cos(\omega_0 t) \quad , \quad y(t) - y_0 = -R\sin(\omega_0 t) \ ; \tag{8.71}$$

As promised, this is a circle centered about (x_0, y_0), with radius R

$$\left(\frac{x - x_0}{R}\right)^2 + \left(\frac{y - y_0}{R}\right)^2 = 1 \ . \tag{8.72}$$

Note that the radius R and initial speed V_y are not independent and we have the relation from (8.69)

$$R = \frac{m\,c\,V_y}{q\,B}\ . \tag{8.73}$$

Hence, the radius of the circle is given by the initial speed in the $x - y$ plane; the larger the B field, the tighter the radius. This means we can use this setup to measure attributes of fundamental charged particles, such as their charge or speed, by measuring the radius of their circular trajectory in known uniform magnetic fields. The bubble chambers of mid-twentieth century were based on this simple principle. Superimposing the x, y motion onto the dynamics in the z direction, we get the celebrated spiral trajectory depicted in Figure 8.6(b) of a charged particle in a uniform magnetic field.

EXAMPLE 8-3: Bubble chamber

The phenomenon of charges circling in uniform magnetic field was one of the first tools used by particle physics to identify and measure properties of sub-atomic particles. Figure 8.7 shows a photograph from a **Bubble Chamber**. Unknown charged particles enter a box immersed in a known external magnetic field. The box is also filled with a dilute fluid that interacts with the particles relatively weakly – creating a trail of bubbles as they pass through. The spirals shown in the figure are actual trajectory of electrons and other more exotic sub-atomic particles! The spiral direction tells us the sign of the charge of the unknown particle. The radius of the spiral can be related to the particle's speed

$$q\,v\,B = m\,\frac{v^2}{r} \Rightarrow \frac{q}{m} B\,r = v \tag{8.74}$$

As the particles moves through the fluid in the box, it loses energy (and hence speed); thus the decreasing radius of the spiral. The device can be used to measure the charge to mass ratio q/m of many sub-atomic particles. This rather simple device spearheaded the golden age of experimental particle physics. Unfortunately, it is rather useless for neutral particles like the π^0 meson...

EXAMPLE 8-4: Ion trapping

Imagine trapping a few ions – or even a single ion – in a small enclosure, poking it around, and observing the intricate physics within it as it reacts to external perturbations. This is

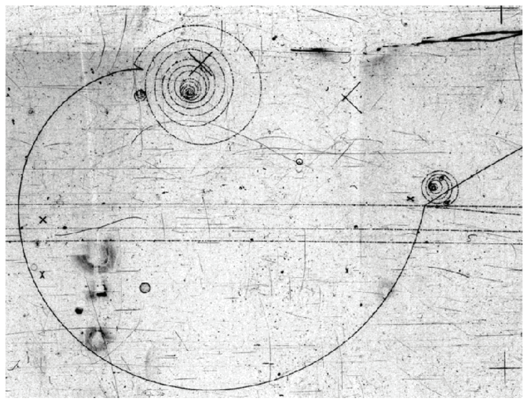


FIG 8.7

something that physicists do regularly using the electromagnetic forces that rule the realm of atomic physics. While many such situations are most interesting because of the quantum physics they allow one to probe, the basic trapping mechanism can be understood using classical physics.

The task is to trap an ion of charge q using external electric and magnetic fields that we can tune arbitrarily. The simplest setup perhaps would involve pure electrostatic fields that we can generate by some arrangement of charges far away from the ion. This is however not the case, as clarified by **Earnshaw's theorem**: it is not possible to construct a stable stationary point for a probe charge using only electrostatic or only magnetostatic fields in vacuum. To see this for the case of electrostatic fields, consider a region of space where the ion probe is to sit and where we have some external electrostatic fields. There are no source charges in this region since these are far away from the trapping region. We then have $\nabla \cdot \boldsymbol{E} = 0$; and using $\boldsymbol{E} = -\nabla\phi$, we get Laplace's equation for the electric potential ϕ

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0\ . \tag{8.75}$$

The potential energy of the ion would then be $U = q\phi$. For trapping the ion, we then need a minimum in this potential. This implies we would need

$$\frac{\partial^2 U}{\partial x^2} > 0 \quad , \quad \frac{\partial^2 U}{\partial y^2} > 0 \quad , \quad \frac{\partial^2 U}{\partial z^2} > 0\ . \tag{8.76}$$

Looking back at Laplace's equation, we see that this is not possible! A saddle surface is the best one can do, and the ion would quickly find a way down the potential, running away to infinity.

There are several ways to circumvent this unfortunate situation. One is to consider time varying electric fields. Imagine a saddle surface that is, say, spinning fast enough that every time the ion ventures a little down the potential, it is quickly pushed back into the middle.

Along this idea, one builds what is know as a **Paul trap**. We leave this case to the Problems section of this chapter. In this example, we discuss instead the so-call **Penning trap**, which involves both electrostatic *and* magnetostatic fields.

The idea of the Penning trap is to start with a uniform magnetic field

$$\boldsymbol{B} = B\hat{\boldsymbol{z}} \tag{8.77}$$

which, as we now know, leads to a spiral trajectory of an ion, circling in the $x-y$ plane with angular speed

$$\omega_0 = \frac{qB}{mc} \ . \tag{8.78}$$

To confine the ion in the z direction as well, we add an electrostatic field described by the electric potential

$$\phi = \frac{\phi_0}{D^2}\left(z^2 - \frac{x^2+y^2}{2}\right) \tag{8.79}$$

where D is some length associated with the geometry of the setup. Note that this electric potential satisfies, as it must, the Laplace equation

$$\nabla^2\phi = 0 \ . \tag{8.80}$$

The Lagrangian for the ion of charge q then becomes

$$L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) - \frac{q\,B}{2\,c}\dot{x}y + \frac{q\,B}{2\,c}\dot{y}x - q\,\frac{\phi_0}{D^2}\left(z^2 - \frac{x^2+y^2}{2}\right) \ . \tag{8.81}$$

It is advantageous to switch to cylindrical coordinates ρ, θ, and z given the symmetries of the potential. We then get

$$L = \frac{1}{2}m\left(\dot{\rho}^2+\rho^2\dot{\theta}^2+\dot{z}^2\right) + \frac{q\,B}{2\,c}\rho^2\dot{\theta} - q\,\frac{\phi_0}{D^2}\left(z^2 - \frac{\rho^2}{2}\right) \ . \tag{8.82}$$

The dynamics in the z direction is then that of a simple harmonic oscillator

$$\ddot{z} = -\omega_z^2 z \tag{8.83}$$

with

$$\omega_z^2 = \frac{2\,q\phi_0}{m\,D^2} \ . \tag{8.84}$$

Note that we would need

$$q\,\phi_0 > 0 \tag{8.85}$$

to make sure the ion is trapped in the z direction. We also can write an energy conservation statement

$$\frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\omega_z^2 z^2 = E_z \tag{8.86}$$

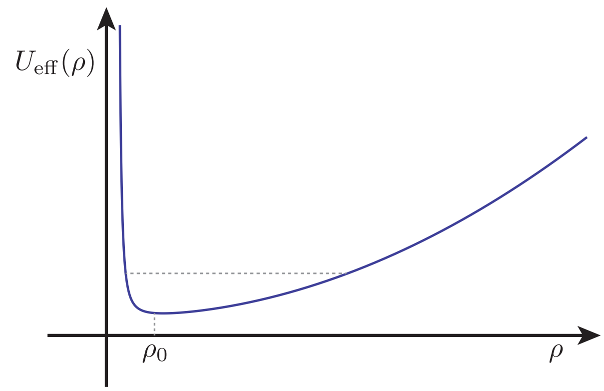


FIG 8.8 : The effective Penning potential. At the minimum, we have a stable circular trajectory. In general however, the radial extent will oscillate with frequency ω_0.

for some constant E_z. To see this, multiply (8.83) by $\dot{z}$ and integrate. For the θ equation of motion, one gets the angular momentum conservation law

$$m\rho^2\dot{\theta} + \frac{qB}{2\,c}\rho^2 = l \tag{8.87}$$

where l denotes the angular momentum constant. Instead of looking at the ρ equation of motion, we realize we have conservation of the Hamiltonian since $\partial L/\partial t = 0$; we then have

$$H = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right) + q\,\frac{\phi_0}{D^2}\left(z^2 - \frac{\rho^2}{2}\right) \tag{8.88}$$

for a constant H. This allows us to write an effective potential for a one dimensional problem in the radial direction ρ – akin to the central force problem we have already seen

$$\frac{1}{2}m\dot{\rho}^2 + U_{\rm eff}(\rho) = H \tag{8.89}$$

$$U_{\rm eff}(\rho) = E_z - \frac{1}{2}l\,\omega_0\,c + \frac{l^2}{2\,m\,\rho^2} + \frac{1}{8}m\rho^2\left(\omega_0^2 - 2\omega_z^2\right) \tag{8.90}$$

where we also eliminated $\dot{\theta}$ in favor of l using (8.87). This potential is known in Figure 8.8. We then identify a minimum at $\rho = \rho_0$

$$\left.\frac{\partial U_{\rm eff}}{\partial\rho}\right|_{\rho_0} = 0 \Rightarrow \rho_0^2 = \frac{2\,l}{m}\left(\omega_0^2 - 2\omega_z^2\right)^{-1/2}\ , \tag{8.91}$$

with the curvature near the minimum given by

$$\left.\frac{\partial^2 U_{\rm eff}}{\partial \rho^2}\right|_{\rho_0} = m\left(\omega_0^2 - 2\,\omega_z^2\right) > 0 \tag{8.92}$$

since typically the oscillation frequency ω_z is much shorter than ω_0

$$\omega_z \ll \omega_0 \ . \tag{8.93}$$

We may then write

$$rho_0 \simeq \sqrt{\frac{2\,l}{m\,\omega_0}} \ . \tag{8.94}$$

At this critical radius, we can look at the angular speed $\dot{\theta}$ using (8.87)

$$\left.\dot{\theta}\right|_{\rho_0} = -\frac{\omega_0}{2} + \frac{1}{2}\sqrt{\omega_0^2 - 2\,\omega_z^2} \simeq -\frac{\omega_z^2}{2\,\omega_0} \equiv -\omega_m \tag{8.95}$$

where in the last step, we used $\omega_z \ll \omega_0$. Hence, $\omega_m \ll \omega_z \ll \omega_0$. The ion circles at ρ_0 very slowly in the $x - y$ plane, while oscillating a little bit faster in the z direction. To see the role of the third frequency ω_0, we note that the general trajectory implied by the effective potential shown in Figure 8.8 involves also radial oscillation. The frequency of this oscillation is given by (8.92)

$$U_{\rm eff}(\rho) \simeq U_{\rm eff}(\rho_0) + \frac{1}{2} m\left(\omega_0^2 - 2\,\omega_z^2\right)\left(\rho - \rho_0\right)^2 \ . \tag{8.96}$$

This means that ρ oscillates with a frequency $\sqrt{\omega_0^2 - 2\,\omega_z^2} \simeq \omega_0$. This is the third fast frequency in the problem, tuned by the strength of the external magnetic field.

The combined motion is shown in Figure 8.9. It involves a slow circular trajectory in the x, y plane of large radius, on top of which we superimpose a slightly faster vertical oscillation in the z direction; and on top of these, we superimpose fast epicycles with tight radii. This setup can in practice achieve ion trapping that last days. But eventually, the configuration is unstable, and other considerations, such as energy leak through electromagnetic radiation, invalidate the analysis.

8.6 Contact forces

Consider a square block of mass M resting on an horizontal floor. Newtonian mechanics tells us that the block experiences two forces: a downward pull $M\,g$ due to gravity,and an upward push by the floor called the *normal force* N. A static scenario implies that $N = M\,g$, so the forces sum to zero and there

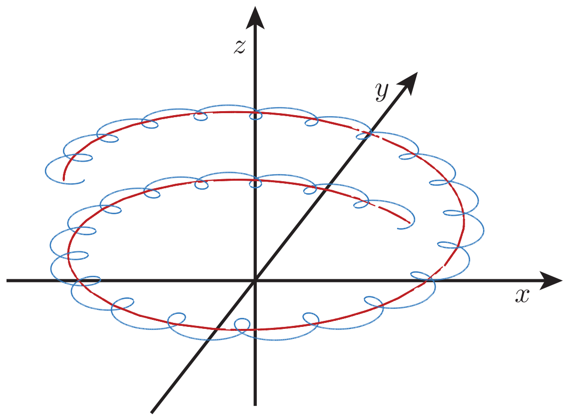


FIG 8.9 : The full trajectory of an ion in a Penning trap. A vertical oscillation along the z axis with frequency ω_z is superimposed onto an fast oscillation of frequency ω_0, while the particle traces a large circle with characteristic frequency ω_m.

is no acceleration. Hence, the normal force adjusts its strength as needed to counteract the gravitational pull $M\,g$ until it matches it. If the balance succeeds, the block stays put on the floor. If however the normal push of the floor is insufficient because the block is too heavy, the floor disintegrates and the block falls through.

The normal force is an example of a **contact force**. It arises by virtue of two objects being in physical contact with each other: in this case, the block and the floor. Contact forces are always electromagnetic in origin. As two entities touch each other, the atoms at the contact interface push against each other through electromagnetic forces. Each of these tiny pushes may be negligible, but with 10^{23} atoms reinforcing the effect, we get a net, effective, *macroscopic* force. Hence, contact forces are not fundamental: they are forces that quantify the effect of many complicated microscopic interactions within a simple, effective, "phenomenological" force law. The normal force, the tension force, and the friction force are examples of such force laws. Underlying all of them we find intricate electromagnetic interactions between large numbers of constituent atoms.

EXAMPLE 8-5: A microscopic model

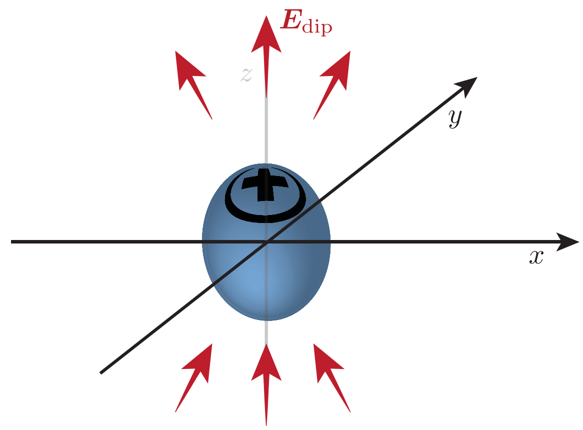


FIG 8.10 : The electric field from a neutral atom leaks out in a dipole pattern due to small asymmetries in the charge distribution of the atom.

In this example, we try to see how the normal force may arise from microscopic electromagnetic interactions at the contact interface between two bodies. We consider the scenario of a block resting on a horizontal floor. The atoms in the block and the floor are in principle electrically neutral. However, small asymmetries in the charge distributions within each atom typically allows for small electric fields to leak out. The leading effect is known as the dipole electric field, as shown in Figure 8.10.

Focus first on the atoms in the floor, arranged in some regular horizontal lattice as shown in Figure 8.11. For simplicity, we consider a single layer of such atoms, right at the block-floor interface;

we also assume for simplicity that the electric dipoles are all nicely aligned as shown. The electric dipole potential from a single atom at (x_0, y_0) is given by

$$\phi_{\mathrm{x_0,y_0}}(x,y,z) = \frac{q\,d\,z}{\left((x-x_0)^2+(y-y_0)^2+z^2\right)^{3/2}} \tag{8.97}$$

where d is the size of an atom; and with the total potential from the entire lattice is then given by

$$\phi_{\mathrm{tot}}(x,y,z) = \sum_{\mathrm{x_0,y_0}} \phi_{\mathrm{x_0,y_0}}(x,y,z) \tag{8.98}$$

where the sum is over all lattice sites (x_0, y_0). If we think of the lattice as an array of squares of size l, we have

$$x_0 = n\,l \quad , \quad y_0 = m\,l \tag{8.99}$$

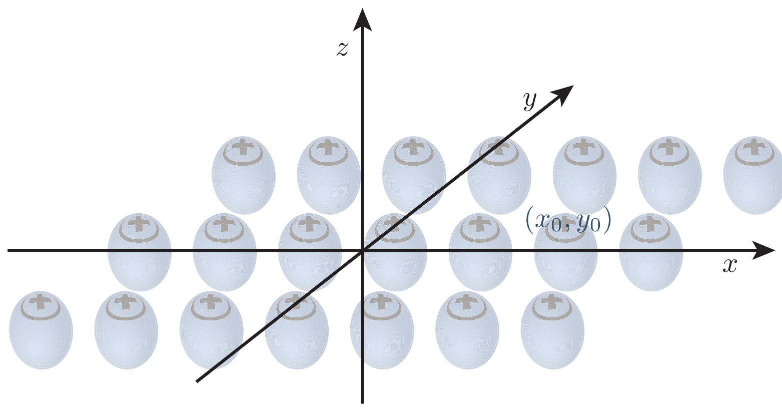


FIG 8.11 : A layer of perfectly aligned dipole at the surface of a floor on which a block is to rest.

with n and m being integers. We then have

$$\phi_{\rm tot}(x,y,z) = \sum_{\rm n,m=-\infty}^{\infty} \frac{q\,d\,z}{\left((x-n\,l)^2+(y-m\,l)^2+z^2\right)^{3/2}} \,. \tag{8.100}$$

We now take the limit where the lattice spacing l is very small compared to any other lengths of interest. Formally, we write

$$l \to 0 \text{ such that } \frac{q\,d}{l^2} \to p \,. \tag{8.101}$$

The idea is that, as we take the lattice spacing to zero, we take $q\,d$ to zero as well so that we have a well-defined macroscopic *density* p, the electric dipole surface density of the floor. p quantifies the relevant macroscopic property of the material that the floor is made of; a quantity that we can presumably measure. We then can turn the discrete sum in (8.100) into an integral, being careful to get the integral measure correct

$$l = \delta n\, l \to \delta x_0 \quad , \quad l = \delta m\, l \to \delta y_0 \,. \tag{8.102}$$

We then can write

$$\begin{aligned} \phi_{\rm tot} &= \sum_{\rm n,m=-\infty}^{\infty} l^2\, q\, d \frac{1}{l^2} \frac{z}{\left((x-n\,l)^2+(y-m\,l)^2+z^2\right)^{3/2}} \\ &= \int_{-\infty}^{\infty} \frac{q\,d}{l^2} dx_0 dy_0 \frac{z}{\left((x-x_0)^2+(y-y_0)^2+z^2\right)^{3/2}} \,. \end{aligned} \tag{8.103}$$

Changing integration variable to $X_0 = x - x_0$ and $Y_0 = y - y_0$, we write the simpler expression

$$\phi_{\rm tot}(z) = p \lim_{\rm L\to\infty} \int_{-\rm L}^{L} dX_0 dY_0 \frac{z}{\left(X_0^2+Y_0^2+z^2\right)^{3/2}} \,. \tag{8.104}$$

Here, we have introduce a cutoff L which may be thought of as the lateral extent of the lattice: eventually, we would want to take L large compared to any other dimensions in the problem. Integrating this, we get

$$\phi_{\rm tot}(z) = 4\,p \lim_{\rm L\to\infty} \cot^{-1}\left(\frac{z\sqrt{2\,L^2+z^2}}{L^2}\right) \; . \tag{8.105}$$

If we want to probe this potential very near the surface of the floor, we can think of expanding in powers of z/L: that is, we look close to the floor surface compared to the lateral extent of the floor – a rather reasonable setup for our purposes. We then get

$$\phi_{\rm tot}(z) = 2\pi\,p - 4\sqrt{2}p\frac{z}{L} + \frac{5\sqrt{2}}{3}p\frac{z^3}{L^3} + \cdots \; . \tag{8.106}$$

The corresponding electric field is

$$E_{\rm tot}(z) = -\frac{\partial\phi_{\rm tot}}{\partial z} = 4\sqrt{2}\frac{p}{L} - 5\sqrt{2}\frac{p}{L}\frac{z^2}{L^2} \; . \tag{8.107}$$

The atoms in the block resting on the floor and located near the contact interface are immersed in this dipole electric field. Using the same microscopic model for the block, the block's atoms near the interface consist of electric dipoles pointing *toward* the floor: the edge of the material in both cases has a slight positive surface charge. Denoting the electric dipole moment per unit area of the block by $\boldsymbol{P}$, the potential energy of the block due to the electric dipole field from the floor is given by

$$U = -L^2\left(\boldsymbol{P}\cdot\boldsymbol{E}\right) = +L^2 P E_{\rm tot}(z) \simeq 4\sqrt{2}P\,p\,L - 5\sqrt{2}P\,p\frac{z^2}{L} \; . \tag{8.108}$$

The total force on the block from this interface interaction would be

$$F_z \simeq -\boldsymbol{\nabla} U = 10\sqrt{2}P\,p\frac{z}{L} \; . \tag{8.109}$$

For a given weight W of the block, a balance may be found at $W = F_z$ at some particular z – distance between block and floor. Notice that this weight is then linear in z: if we were to double it from say 100 N to 200 N, z would also double perhaps from ten Angstroms to twenty, which is still a negligible change microscopically. That is, we have $\delta W/W = \delta z/z$. Note that this conclusion is only valid for small $z \ll L$.

As demonstrated through the last example, contact forces trace their microscopic origin to electromagnetic forces between constituent atoms and molecules. In the simplest systems, it may be possible to model the complex microscopic interactions as simple effective constraint rules at the macroscopic level.

For example, a square block resting on the floor experiences a single upward push from the floor called the normal force. The magnitude of this normal force is such that the block does not fall through the floor – as long as the floor is strong enough and the box light enough. Hence, the effect of the normal force is to constraint the vertical motion of the box: it can only move sideways and not up and down. The dynamics of the center of mass of the box is now reduced to two degrees of freedom, from the original three. Similarly, the tension force in the rope of a blob pendulum implements a constraint that assures that the ball at the end of the rope does not fly way beyond a distance equal to the length of the rope. Many more such microscopic electromagnetic effects translate into statements of constraints on the degrees of freedom. In this section, we develop the technology of solving mechanics problem – with various types of constraints.

Consider a mechanical system parameterized by N coordinates q_k, $k = 1 \ldots N$. However, due to some constraint forces, there are P algebraic relations amongst these coordinates, given by

$$C_l(q_1, q_2, \cdots, q_N, t) = 0 \tag{8.110}$$

where $l = 1 \cdots P$. This means we have effectively $N - P$ generalized coordinates or degrees of freedom, instead of the usual N. For example, if we take the simplest example of a block of mass m resting on a horizontal floor, we start with $N = 3$ coordinates x, y, z; then we specify $P = 1$ constraint, the equation $z = 0$, where z is the vertical direction and the center of mass of the block rests at $z = 0$. At this point, we have two choices. We could try to use the P relations (8.110) to eliminate P q_k's, and write the Lagrangian in terms of $N - P$ generalized coordinates. This is in a sense what we have been doing so far. For the example at hand, we would write

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - m\, g\, z = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2\right) \ . \tag{8.111}$$

Alternatively, we may want to delay the constraint implementation. There are two good reasons for this. First, it may be difficult or inconvenient to eliminate P q_ks using the constraints (8.110). Second, we may be interested in finding the *constraint forces* underlying these constraint relations. For example, we may want to find out the normal force on a block sliding along a curved rail; when this force vanishes, the block is losing contact with the rail, and we would be able to determine this critical point in the evolution.

Hence, we now focus on a method that delays implementing the constraints in a mechanical problem, instead dealing with all N q_k's in the problem. The complication arises because the variational formalism assumes *independent* q_ks: the q_ks must not have relations amongst them. Otherwise, in the process of extremizing the action, we would get to the step

$$\delta I = \int \left(\frac{d}{dt} \left(-\frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial L}{\partial q_k} \right) \delta q_k dt = 0 \ , \tag{8.112}$$

and, since the δq_k's are not independent due to (8.110), we cannot conclude that the parenthesized expression – that is the Lagrange equations of motion – are necessarily satisfied individually for every k.

Instead, let us consider the new Lagrangian defined as

$$L' = L + \sum_{l=1}^{P} \lambda_l C_l \tag{8.113}$$

where we have introduced P additional degrees of freedom labeled λ_l with $l = 1 \cdots P$, each multiplying a related constraint equation from (8.110). For example, with the block on a horizontal floor example, we would write

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) - m\, g\, z + \lambda_1 z \ . \tag{8.114}$$

We now assume that the constraint equations (8.110) are *not* satisfied a priori. We then have $N+P$ degrees of freedom: N q_k's, and P λ_l's. Correspondingly, we have $N + P$ equations of motion. For the example at hand, we have $N + P = 3 + 1 = 4$ variables left: x, y, z, and λ_1. The equations of motion are

- P equations for the λ_l's

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{\lambda}_l} \right) - \frac{\partial L'}{\partial \lambda_l} = 0 \Rightarrow \frac{\partial L'}{\partial \lambda_l} = 0 \Rightarrow C_l = 0 \ . \tag{8.115}$$

  These are simply the original constraints (8.110)! But they now arise dynamically through the equations of motion and need not be implemented from the outset. The P parameters labeled λ_l are called **Lagrange multipliers**. For our simple example, we have one Lagrange multiplier λ_1 with equation of motion $z = 0$ as needed.

- N equations for the q_k's

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}_k} \right) - \frac{\partial L'}{\partial q_k} = 0 \ . \tag{8.116}$$

In terms of the original L, these look like

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} - \lambda_l \frac{\partial C_l}{\partial q_k} = 0 \ . \tag{8.117}$$

Note the additional terms involving the λ_l's (l is repeated and hence summed over). We can rewrite these as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \mathcal{F}_k = \lambda_l \frac{\partial C_l}{\partial q_k} \tag{8.118}$$

where we defined the **generalized constraint forces** $\mathcal{F}_k$ that essentially enforce the constraints onto the q_k dynamics. For the block on the horizontal floor problem, these become

$$m\,\ddot{x} = 0 \quad , \quad m\,\ddot{y} = 0 \quad , \quad m\,\ddot{z} = \lambda_1 \ . \tag{8.119}$$

How do we relate the generalized constraint forces – and hence the λ's – to the actual forces? For every object i in the problem located at position $\boldsymbol{r}_i$, denote the total constraint force acting on it by $\boldsymbol{F}_i$. We also know the relations $\boldsymbol{r}_i(q_k, t)$ that connect the position of every object to the generalized coordinates q_k. Using all this, we can relate the Lagrange multipliers to the constraint forces by noting that the right hand side of the new Lagrange equations must be given by

$$\mathcal{F}_k = \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k} = \lambda_l \frac{\partial C_l}{\partial q_k} \ . \tag{8.120}$$

The easiest way to see this is to start with $\boldsymbol{F}_i = -\boldsymbol{\nabla}_i U$ for every particle i, and apply the chain rule to write it in terms of $\partial U/\partial q_k = \mathcal{F}_k$. Once we determine the Lagrange multipliers λ_l, we can use this relation to read off constraint forces $\boldsymbol{F}_i$. For example, for the example at hand, we have

$$\lambda_1 = \boldsymbol{F} \cdot \frac{\partial \boldsymbol{r}}{\partial z} = F_z = N \ . \tag{8.121}$$

Hence λ_1 *is* the expected vertical normal force.

Before we apply this general and abstract treatment to particular examples, we note that this method of constraints can easily be generalized a bit further. Looking back at (8.112), we see that we only need that the constraint is in the form of a *variation*. That is, we can relax the constraint relations between the δq_k as long as the constraint can be added into the infinitesimal change of the action in the form

$$\delta I = \int \left(\frac{d}{dt} \left(-\frac{\partial L}{\partial \dot{q}_k} \right) + \frac{\partial L}{\partial q_k} + \lambda_l a_{\rm lk} \right) \delta q_k dt = 0 \ , \tag{8.122}$$

for some functions $a_{\rm lk}$ of q_k and t. That is, if the constraints on the generalized coordinates q_k can be written in the form

$$a_{\rm lk} \delta q_k + a_{\rm lt} \delta t = 0 \tag{8.123}$$

where $a_{\rm lk}$ and $a_{\rm lt}$ are arbitrary functions of q_k and t, we can write the following set of $N + P$ equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \lambda_l a_{\rm lk} \tag{8.124}$$

$$a_{\rm lk} \dot{q}_k + a_{\rm lt} = 0 \ . \tag{8.125}$$

This is a useful generalization of the original formulation of the problem given by (8.115)) and (8.118 because we do not necessarily have

$$\frac{\partial a_{\rm lk}}{\partial t} \neq \frac{\partial a_{\rm lt}}{\partial q_k} \ . \tag{8.126}$$

If the constraints could be written as before in terms of P *algebraic* relations $C_l(t, q) = 0$, we could write

$$0 = dC_l = \frac{\partial C_l}{\partial q_k} dq_k + \frac{\partial C_l}{\partial t} dt = a_{\rm lk} dq_k + a_{\rm lt} dt \tag{8.127}$$

Reading off the needed $a_{\rm lk}$ and $a_{\rm lt}$ as functions of q_k and t. However, in this special class of constraints, we *would* have

$$\frac{\partial a_{\rm lk}}{\partial t} = \frac{\partial a_{\rm lt}}{\partial q_k} \tag{8.128}$$

because of the commutativity of derivatives.

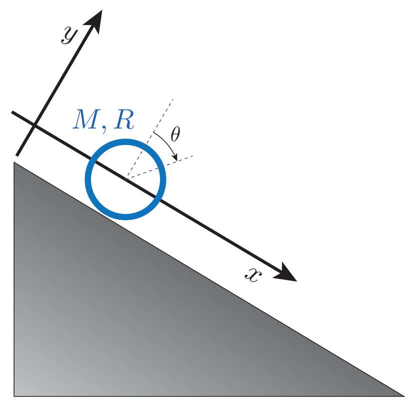


FIG 8.12 : A hoop rolling down an inclined plane without slipping.

Hence, to summarize, when dealing with a mechanics problem involving constraints of the form (8.123)), we may choose to delay the implementation of the constraints in an effort to extract constraint forces acting on the system. To do so, we would need to solve a set of $N+P$ differential equations given by (8.124) and (8.125). In addition to finding $q_k(t)$, this procedure also leads to P Lagrange multipliers that can be related to constraint forces through (8.120. The best way to learn this technology is once again through examples.

EXAMPLE 8-6: Rolling down the plane

Consider a hoop of radius R and mass M rolling down an inclined plane as shown in Figure 8.12. To describe the hoop, we may prescribe three variables: a position in two dimensions $\boldsymbol{r}$ and rotational angle θ

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} \quad , \quad \theta \tag{8.129}$$

where the coordinate system is set up tilted as shown in the figure for convenience. However, we know of two potential constraints. First, the hoop is prevented from falling through the incline because of the normal force. This contact force enforces the constraint

$$y = 0 \Rightarrow dy = 0 \ . \tag{8.130}$$

Furthermore, if there is friction involved, the hoop may roll down without slipping. The frictional contact force enforces the constraint

$$R\, d\theta = dx \tag{8.131}$$

At the end of the day, these two constraint suggest that the problem involves only *one* degree of freedom, not three. The kinetic energy is

$$T = \frac{1}{2} M \left(\dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} M R^2 \dot{\theta}^2 \tag{8.132}$$

while the potential energy is

$$V = -M g x \sin\varphi \ . \tag{8.133}$$

We may then proceed as usual by implementing the constraint from the outset. This leads to the Lagrangian with one degree of freedom, which we may choose to be the x coordinate

$$L = T - V = M \dot{x}^2 + M g x \sin\varphi \ . \tag{8.134}$$

The equation of motion then tells us the acceleration down the incline is

$$\ddot{x} = \frac{g}{2} \sin\varphi \ . \tag{8.135}$$

What if we are interested in knowing the size of the friction force? We still do not care about the normal force. This means we will implement the normal force constraint given by (8.130)) from the outset, eliminating the y coordinate; however, we will delay implementing the frictional constraint given by (8.131). This leaves us with two of the original coordinates, x and θ; and, a new degree of freedom, λ_1, a Lagrange multiplier that will measure the size of the friction force. We then will need three differential equations. Looking back at (8.131) and mapping it onto the general form (8.125, we read off

$$a_{1\theta} = R \quad , \quad a_{1\mathrm{x}} = -1 \quad , \quad a_{1\mathrm{t}} = 0 \ . \tag{8.136}$$

We then write the Lagrangian in terms of x and θ only

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} M R^2 \dot{\theta}^2 + M g x \sin\varphi \ ; \tag{8.137}$$

and use the modified equations of motion given by (8.124). This gives for the x direction

$$M \ddot{x} - M g \sin\varphi = -\lambda_1 = \mathcal{F}_x \ . \tag{8.138}$$

And for the θ direction, one gets

$$M R^2 \ddot{\theta} = \lambda_1 R = \mathcal{F}_\theta \tag{8.139}$$

We now have a set of three differential equations given by (8.138)), (8.139, and

$$R \dot{\theta} = \dot{x} \tag{8.140}$$

which follows from the constraint (8.131). Solving this system of equations, we get

$$\ddot{x} = \frac{g}{2} \sin\varphi \quad , \quad \lambda_1 = \frac{M g}{2} \sin\varphi \ . \tag{8.141}$$

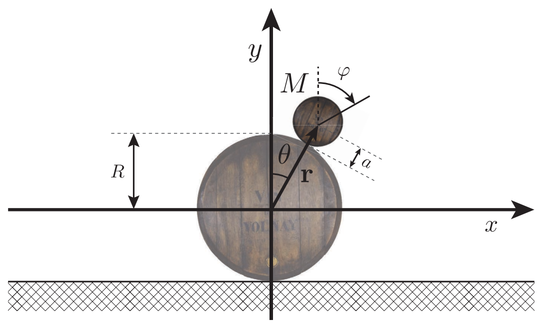


FIG 8.13 : Two barrels stacked on top of each other. The lower barrel is stationary, while the upper one rolls down without slipping.

The novelty is of course the determination of λ_1, which we can now relate to the friction force by

$$\mathcal{F}_\theta = \lambda_1 R = \frac{MgR}{2}\sin\varphi = \boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial\theta} = \boldsymbol{F}\cdot\frac{\partial x}{\partial\theta}\hat{\boldsymbol{x}} = F_x R = \tau\ . \tag{8.142}$$

That is, the friction force $M\,g\,\sin\varphi/2$ applies a torque equal to $M\,g\,R\,\sin\varphi/2$. Hence, the method of Lagrangian multipliers allowed us to selectively extract forces of constraint in a mechanical problem without abandoning the powerful and elegant machinery of the Lagrangian formalism.

EXAMPLE 8-7: Stacking barrels

Consider the problem of two cylindrical barrels on top of each other, as shown in Figure 8.13. The bottom barrel is fixed in position and orientation, but the top one, of mass M, is free to move. It starts slipping off from its initial position at the top, rolling down without slipping due to friction between the barrels. The problem is to find the point along the lower barrel where the top barrel loses contact with it. That is, we need to find out the moment when the normal force acting on the top barrel vanishes.

We are tracking the motion of the top barrel. Hence, we have a priori three variables to keep track of, two positions r and θ, and one rotational angle φ

$$\boldsymbol{r} = r\hat{\boldsymbol{r}} \quad , \quad \varphi \tag{8.143}$$

where we use polar coordinates centered on the bottom barrel to track the position of the top barrel. A normal force acting on the top barrel enforces the constraint

$$R + a = r \ . \tag{8.144}$$

If the top barrel rolls without slipping, the friction force enforces the constraint

$$a d\varphi = R d\theta \tag{8.145}$$

The full kinetic energy in terms of r, θ, φ is then

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m a^2 \dot{\varphi}^2 \tag{8.146}$$

with potential energy

$$V = m\,g\,r\,\cos\theta \ . \tag{8.147}$$

Note that we included the rotational kinetic energy of the hoop given by $(1/2) m\, a^2 \dot{\varphi}^2$. Since we are only interested in the normal force, we implement the frictional constraint (8.145) from the outset to eliminate φ in favor of θ. We then get

$$L = T - V = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} m R^2 \dot{\theta}^2 - m\,g\,r\,\cos\theta \tag{8.148}$$

We do *not* implement the normal force constraint (8.144), which we now write in canonical form

$$dr = 0 \ . \tag{8.149}$$

This means we will have three degrees of freedom: r, θ, and a Lagrange multiplier λ_1 associated with (8.144)). We can now read off the relevant coefficients of (8.125

$$a_{1\mathrm{r}} = 1 \qquad a_{1\theta} = a_{1\mathrm{t}} = 0 \ . \tag{8.150}$$

The equations of motion follow from (8.124). For the r direction, we get

$$m\ddot{r} - m r \dot{\theta}^2 + m g \cos\theta = \lambda_1 = \mathcal{F}_r = \boldsymbol{F} \cdot \frac{\partial \boldsymbol{r}}{\partial r} = F_r = N \tag{8.151}$$

while for the θ direction, we get

$$m\,r^2 \ddot{\theta} + m\,R^2 \ddot{\theta} - m g r \sin\theta = 0 \tag{8.152}$$

The third and final equation follows from the constraint (8.144), which we now write as

$$\dot{r} = 0 \ . \tag{8.153}$$

We may now proceed solving this set of three differential equations. It is however slightly more elegant to realize that we do have energy conservation in this problem. Hence, we can write

$$E = T + V = \frac{1}{2} m \left((R + a)^2 \dot{\theta}^2 \right) + \frac{1}{2} m R^2 \dot{\theta}^2 + m g (R + a) \cos\theta \tag{8.154}$$

with $r = R + a$ from the constraint and E a constant. Arranging for the initial conditions $\theta(0) = 0$ and $\dot{\theta}(0) = 0$, we have $E = m\,g\,(R + a)$. This implies

$$\dot{\theta}^2 = \frac{2g\,(R+a)}{2\,R^2 + a^2 + 2Ra}\,(1 - \cos\theta)\ . \tag{8.155}$$

From (8.151), we have

$$-m\,(R+a)\,\dot{\theta}^2 + m\,g\,\cos\theta = \lambda_1 = N\ . \tag{8.156}$$

And setting $N = 0$ at the moment when the top barrel loses contact with the bottom one, we get

$$\dot{\theta}_c^2 = \frac{g\cos\theta_c}{R+a} \tag{8.157}$$

where θ_c denotes the critical angle at which this condition is satisfied. Using (8.155), we find

$$\cos\theta_c = \frac{2}{3 + \frac{R^2}{(R+a)^2}}\ . \tag{8.158}$$

There are two interesting limiting cases within this expression. If the top barrel is tiny, we have $a/R \ll 1$, which leads to

$$\cos\theta_c = \frac{1}{2}\ . \tag{8.159}$$

On the other hand, taking the opposite regime $a/R \gg 1$, one gets

$$\cos\theta_c = \frac{2}{3}\ . \tag{8.160}$$

EXAMPLE 8-8: On the rope

A classic problem is that of a simple pendulum consisting of a point mass m at the end of a rope of length l swinging in a plane (see Figure 8.14). We would like to determine the tension in the rope as a function of the angle θ. We start with two variables

$$\boldsymbol{r} = r\hat{\boldsymbol{r}} \tag{8.161}$$

using a polar coordinate system centered at the pivot as shown in the figure. Hence, our variables are r and θ. However, we have a constraint enforced by the tension in the rope

$$r = l \Rightarrow dr = 0\ \Rightarrow\ a_{1\mathrm{r}} = 1\quad,\quad a_{1\mathrm{t}} = 0 \tag{8.162}$$

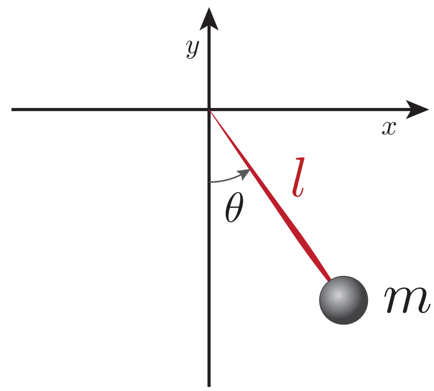


FIG 8.14 : A pendulum with a single constraint given by the fixed length of the rope.

which we immediately used to read off the relevant coefficients for (8.125). The kinetic energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{8.163}$$

while the potential energy is simply

$$V = -mgr\cos\theta \ . \tag{8.164}$$

The Lagrangian becomes

$$L = T - V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + mgr\cos\theta \tag{8.165}$$

where we keep track of both r and θ as independent degrees of freedom at the cost of introducing a single Lagrange multiplier λ_1 associated with the constraint. The equation of motion for r comes from (8.124) and looks like

$$m\ddot{r} + mr\dot{\theta}^2 - mg\cos\theta = \lambda_1 = \boldsymbol{F}\cdot\frac{\partial \boldsymbol{r}}{\partial r} = F_r = T \ . \tag{8.166}$$

While that for θ looks like

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) + mgr\sin\theta = 0 \ . \tag{8.167}$$

The three degrees of freedom are associated with three differential equations, where the third comes of course from the constraint (8.162), which we now write as

$$\dot{r} = 0 \ . \tag{8.168}$$

This is sufficient to determine all three variables of interest. Once again, however, it is easier to use energy conservation

$$E = T + V = \frac{1}{2} m\, l^2 \dot{\theta}^2 - m\, g\, l\, \cos\theta \ . \tag{8.169}$$

We start with initial conditions

$$\theta(0) = \theta_0 \quad , \quad \dot{\theta}(0) = 0 \ \Rightarrow \ E = -m\, g\, l\, \cos\theta_0 \ . \tag{8.170}$$

We then have

$$\dot{\theta}^2 = \frac{2g}{l}\left(\cos\theta - \cos\theta_0\right) \ . \tag{8.171}$$

This allows us to find the Lagrange multiplier λ_1 in terms of θ

$$\lambda_1 = -mg\cos\theta + 2mg\left(\cos\theta - \cos\theta_0\right) = -mg\left(\cos\theta - 2\,\cos\theta_0\right) = T \ , \tag{8.172}$$

the tension in the rope.

Problems

PROBLEM 8-1: Consider an infinite wire carrying a constant linear charge density λ_0. Write the Lagrangian of a probe charge Q in the vicinity, and find its trajectory.

PROBLEM 8-2: Consider the oscillating Paul trap potential

$$U(z,r) = \frac{U_0 + U_1 \cos \Omega t}{r_0^2 + 2\, z_0^2} \left(2\, z^2 + \left(r_0^2 - r^2\right)\right) \tag{8.173}$$

written in cylindrical coordinates. (a) Show that this potential satisfies Laplace's equation. (b) Consider a point particle of charge Q in this potential. Analyze the dynamics using a Lagrangian and show that the particle is trapped.

PROBLEM 8-3: Show that the Coulomb gauge $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$ is a consistent gauge condition.

PROBLEM 8-4: Find the residual gauge freedom in the Coulomb gauge.

PROBLEM 8-5: Show that the Lorentz gauge $\partial_\mu A_\nu \eta_{\mu\nu} = 0$ is a consistent gauge condition.

PROBLEM 8-6: Find the residual gauge freedom in the Lorentz gauge.

PROBLEM 8-7: A particle of mass m slides inside a smooth hemispherical bowl of radius R. Use spherical coordinates r, θ and φ to describe the dynamics. (a) Write the Lagrangian in generalized coordinates and solve the dynamics. (b) Repeat the exercise using a Lagrange multiplier. What does the multiplier measure in this case?

PROBLEM 8-8: A pendulum consisting of a ball at the end of a rope swings back and forth in a two-dimensional vertical plane, with the angle θ between the rope and the vertical evolving in time. However, the rope is pulled upward at a constant rate so that the length l of the pendulum's arm is decreasing as in $dl/dt = -\alpha \equiv$ constant. (a) Find the Lagrangian for the system with respect to the angle θ. (b) Write the corresponding equations of motion. (c) Repeat parts (a) and (b) using Lagrange multipliers.

PROBLEM 8-9: A particle of mass m slides inside a smooth paraboloid of revolution whose surface is defined by $z = \alpha\rho^2$, where z and ρ are cylindrical coordinates. (a) Write the Lagrangian for the three dimensional system using the method of Lagrange multipliers. (b) Find the equations of motion.

PROBLEM 8-10: In certain situations, it is possible to incorporate frictional effects without introducing the dissipation function. As an example, consider the Lagrangian

$$L = e^{\gamma t} \left(\frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2\right) \; . \tag{8.174}$$

(a) Find the equation of motion for the system. (b) Do a coordinate change $s = e^{\gamma t/2} q$. Rewrite the dynamics in terms of s. (c) How would you describe the system?

PROBLEM 8-11: A massive particle moves under the acceleration of gravity and without friction on the surface of an inverted cone of revolution with half angle α. (a) Find the Lagrangian in polar coordinates. (b) Provide a complete analysis of the trajectory problem, mimicking what we did for the case of Newtonian potential in class. Do not integrate the final orbit equation, but explore circular orbits in detail. (c) Repeat using Lagrange multipliers.

PROBLEM 8-12: A toy model for our expanding universe during the inflationary epoch consists of a circle of radius $r(t) = r_0 e^{\omega t}$ with our miserable lives confined on the one-dimensional world that is the circle. To probe the physics, imagine two point masses of identical mass m free to move on this circle without friction, connected by a spring of spring constant k and relaxed length zero, as depicted in the figure. (a) Write the Lagrangian for the two-particle system in terms of the common radial coordinate r and the two polar coordinates θ_1 and θ_2. Do NOT implement the radial constraint $r(t) = r_0 e^{\omega t}$ yet. (b) Using a Lagrange multiplier for the radial constraint, write *four* equations describing the dynamics. In process, you better show that

$$a_{1r} = 1 \quad , \quad a_{1t} = -\omega r \ . \tag{8.175}$$

(c) Consider the coordinate relabeling

$$\alpha \equiv \theta_1 + \theta_2 \quad , \quad \beta \equiv \theta_1 - \theta_2 \ . \tag{8.176}$$

Show that the equations of motion of part (b) for the two angle variables θ_1 and θ_2 can be rewritten in a decoupled form as

$$\ddot{\alpha} = C_1 \dot{\alpha} \ ; \tag{8.177}$$

$$\ddot{\beta} = C_2 \dot{\beta} + C_3 \beta \ ; \tag{8.178}$$

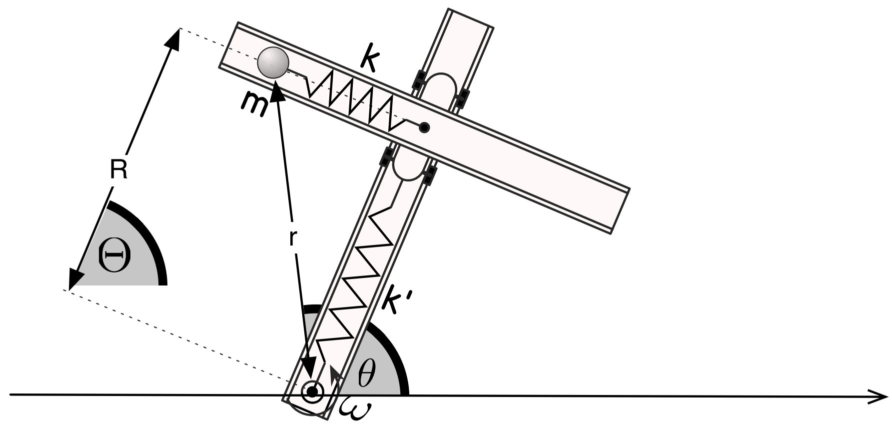
where C_1, C_2 and C_3 are constants that you will need to find.

PROBLEM 8-13: For the previous problem, (a) identify a symmetry transformation $\{\delta t, \delta\alpha, \delta\beta\}$ for this system. Find the associated conserved quantity. What would you call this conserved quantity? (b) Then find $\alpha(t)$ and $\beta(t)$ using equations (8.177) and (8.178). Use the boundary conditions:

$$\alpha(0) = \alpha_0 \quad , \quad \dot{\alpha}(0) = 0 \quad , \quad \beta(0) = 0 \quad , \quad \dot{\beta}(0) = C \quad , \tag{8.179}$$

What is the effect of the expansion on the dynamics? NOTE: This conclusion is the same as in the more realistic three dimensional cosmological scenario! (c) Find the force on the particles exerted by the expansion of the universe. Write this as a function of $\alpha(t)$, $\beta(t)$, and $r(t)$; and then show that its limiting form for later times in the evolution is given by

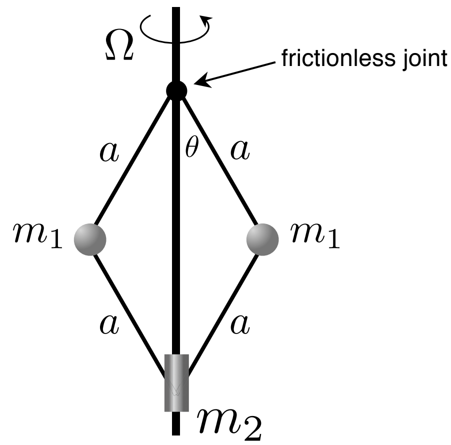
$$2\, m\, \omega^2\, r(t) \ . \tag{8.180}$$



PROBLEM 8-14:

The figure above shows a mass m connected to a spring of spring constant k along a wooden track. The mass is restricted to move along this track without friction. The whole setup is mounted on a toy wagon of zero mass resting on a track along a second frictionless beam. The wagon is connected by a spring of spring constant k' to an axle about which the whole darn thing is spinning with constant angular speed ω. The figure is a top-down view, with gravity pointing into the page, and the rest length of each spring is zero. Take a deep breath while reciting the Latin alphabet backwards. (a) First, write the Lagrangian of the system in terms of the four variables r, θ, R, and Θ shown on the Figure, without implementing any constraints. (b) Identify two constraint equations. Implement the one keeping the two tracks perpendicular to each other into the result of part (a) by eliminating R. Do NOT implement the one that spins things at constant angular speed ω. (c) Introducing a Lagrange multiplier for the constraint having to do with the spin, write four differential equations describing the system. (d) Identify the force on the mass m due to the spin of the setup, and find all conditions for which this force vanishes!

PROBLEM 8-15: Consider the system shown in the figure below. The particle of mass m_2 moves on a vertical axis without friction and the whole contraption rotates about this axis with a constant angular speed Ω. The frictionless joint near the top assures that the three masses always lie in the same plane; and the rods of length a are all rigid. (HINT: Use the origin of your coordinate system at the upper frictionless joint.) (a) Find the equation of motion in terms of one degree of freedom θ. (b) Using the method of Lagrange multipliers, find the torque on the masses m_1 due to the rotational motion. (c) Find a static solution



in θ and identify the corresponding angle in terms of m_1, m_2, g, a, and Ω. Consider some limits/inequalities in your result and comment on whether they make sense. (d) Is the solution in part (c) stable? if so, what is the frequency of small oscillations about the configuration. (HINT: Use $\xi = \cos\theta$ and work on the Lagrangian instead of the equation of motion.)