Problem 1 – Townsend 7.4 Use $\hat{a} |0\rangle = 0$ and therefore $\langle p | \hat{a} | 0 \rangle = 0$ to solve directly for $\langle p | 0 \rangle$, the ground state wave function of the harmonic oscillator in momentum space. Normalize the wave function. Hint: Recall the result of Problem 6.2,

$$\langle p | x | \psi \rangle = i \hbar \frac{\partial}{\partial p} \langle p | \psi \rangle$$

Problem 2 – Townsend 7.10 Prove that the parity operator $\hat{\Pi}$ is Hermitian.

Problem 3 – Townsend 7.12 Calculate the probability that a particle in the ground state of the harmonic oscillator is located in a classically disallowed region, namely, where $V(x) > E$. Obtain a numerical value for the probability. Suggestion: Express your integral in terms of a dimensionless variable and compare with the tabulated values of the error function. [Or just use a program like Maple.]

Problem 4 Many short-pulse lasers have a pulse-shaping mechanism that produces “sech$^2$” pulses of the form

$$I(t) \propto I_0 \left[ \cosh(t/\tau) \right]^{-2}$$

The proportionality constant can be determined by normalization. The intensity is proportional to the square of the electric field, which is the dynamical variable that appears in the equations of motion. Therefore, the quantum mechanical state of the photon in such a pulse is proportional to the hyperbolic secant.

In the vacuum, all frequency components of the light wave travel at the same speed $c$. Thus, we will make the following identification:

$$\langle x | \psi \rangle = \psi(x) = A \text{sech}(x/\sigma)$$

where $x = ct$ and $\sigma = c\tau$.

Since for massless particles, $\omega = ck$, the Heisenberg uncertainty relation for position and momentum of photons corresponds to the time-energy uncertainty relation.

(a) What is $\langle p | \psi \rangle$? Hint: the Fourier transform of sech $x$ is $\sqrt{\pi/2} \text{sech}(\pi k/2)$. Since the momentum is linearly related to the energy for massless particles, you have now determined the frequency content of the pulse.

(b) Make a plot of the probability density of the photon vs. momentum (energy).

(c) Compute the time-bandwidth product. That is, compute $\Delta t \Delta \omega$. Is it consistent with the Heisenberg uncertainty principle? Hint: $\int_{-\infty}^{\infty} x^2 \text{sech}^2 x \, dx = \pi^2/6$, as can be shown via contour integration.