Homework Assignment 2, due 27 January 2006

Problem 1 – Townsend 2.1  
Show that
\[ \lim_{N \to \infty} \left( 1 + \frac{x}{N} \right)^N = e^x \]
by comparing the Taylor series expansions for the two functions.

Problem 2 – Townsend 2.2  
Use Dirac notation (the properties of kets, bras, and inner products) directly without explicitly using matrix representations to establish that the projection operator \( \hat{P}_+ \) is Hermitian. Use the fact that \( \hat{P}_+^2 = \hat{P}_+ \) to establish that the eigenvalues of the projection operator are 1 and 0.

Problem 3 – Townsend 2.3  
Determine the matrix representation of the rotation operator \( \hat{R}(\phi \hat{k}) \) using the states \( |+z\rangle \) and \( |-z\rangle \) as a basis. Using your matrix representation, verify that the rotation operator is unitary, that is that it satisfies \( \hat{R}^\dagger(\phi \hat{k}) \hat{R}(\phi \hat{k}) = 1 \).

Problem 4 – Townsend 2.4  
Determine the column vectors representing the states \( |+x\rangle \) and \( |-x\rangle \) using the states \( |+y\rangle \) and \( |-y\rangle \) as a basis.

Problem 5 – Townsend 2.5  
What is the matrix representation of \( \hat{J}_z \) using the states \( |+y\rangle \) and \( |-y\rangle \) as a basis? Use this representation to evaluate the expectation value of \( S_z \) for a collection of particles each in the state \( |-y\rangle \).

Problem 6 – Townsend 2.6  
Evaluate \( \hat{R}(\theta \hat{j}) |+z\rangle \), where \( \hat{R}(\theta \hat{j}) = e^{-i \hat{J}_y \theta/\hbar} \) is the operator that rotates kets counterclockwise by an angle \( \theta \) about the \( y \) axis. Show that \( \hat{R}(\frac{\pi}{2} \hat{j}) |+z\rangle = |+x\rangle \).

Problem 7 – Townsend 2.8  
A photon polarization state for a photon propagating in the \( z \) direction is given by
\[ |\psi\rangle = \frac{\sqrt{2}}{3} |x\rangle + \frac{i}{\sqrt{3}} |y\rangle \]

(a) What is the probability that a photon in this state will pass through a perfect polarizer with its transmission axis oriented in the \( y \) direction?

(b) What is the probability that a photon in this state will pass through a perfect polarizer with its transmission axis \( y' \) making an angle \( \phi \) with the \( y \) axis?

(c) A beam carrying \( N \) photons per second, each in the state \( |\psi\rangle \), is totally absorbed by a black disk with its normal to the surface in the \( z \) direction. How large is the torque exerted on the disk? In which direction does the disk rotate? Reminder: The photon states \( |R\rangle \) and \( |L\rangle \) each carry a unit \( \hbar \) of angular momentum parallel and antiparallel, respectively, to the direction of propagation of the photons.

(d) How would the result for each of these questions differ if the polarization state were
\[ |\psi'\rangle = \sqrt{\frac{2}{3}} |x\rangle + \frac{1}{\sqrt{3}} |y\rangle \]
that is, the “\( i \)” in the state \( |\psi\rangle \) is absent?
Problem 8 – Townsend 2.11  Determine the matrix representation of the angular momentum operator $\hat{J}_z$ using both the circular polarization vectors $|R\rangle$ and $|L\rangle$ and the linear polarization vectors $|x\rangle$ and $|y\rangle$ as a basis. Check that the operator is Hermitian in both matrix representations.

Problem 9 – Townsend 2.18  A beam of linearly polarized light is incident on a quarter-wave plate with its direction of polarization oriented at $30^\circ$ to the optic axis. Subsequently, the beam is absorbed by a black disk. Determine the rate angular momentum is transferred to the disk, assuming the beam carries $N$ photons per second.