Problem Set 1 Due: 6 Sept 2006

Please staple problems 1+2 and 3+4.

1. Common Birthdays

(a) Calculate the probability that in a group of $N$ people two or more have the same birthday. You can disregard leap years.

(b) How large a crowd do you need to make the probability found in (a) equal 50%? Please explain how you find your answer.

(c) For a group of 100 people, compute the probability that all of them have different birthdays. Use the Stirling approximation to find a numerical answer.

2. Accessible Microstates (Schroeder, problem 2.23)

Consider a two-state paramagnet with $10^{23}$ elementary dipoles, with the total energy fixed at zero so that exactly half the dipoles point up and half point down.

(a) How many microstates are “accessible” to this system? Please evaluate your answer using the Stirling approximation.

(b) Suppose that the microstate of this system changes $10^9$ times a second. How many microstates will it explore in $10^{10}$ years (roughly the age of the Universe)?

(c) Is it correct to say that, if you wait long enough, a system will eventually be found in every “accessible” microstate? Explain your answer, and discuss the meaning of the word “accessible”.

(d) Discuss the validity of the ergodic assumption in light of your results.

3. Poisson Distribution (Reif, 1.8)

In section we showed that the probability $P(N, n)$ that an event characterized by a probability $p$ occurs $n$ times in $N$ trials is given by the binomial distribution:

$$P(N, n) = \frac{N!}{n!(N-n)!}p^n(1-p)^{N-n}. \quad (1)$$

Consider a situation where the probability $p$ is small ($p \ll 1$) and where one is interested in the case $n \ll N$. (Note that if $N$ is large, $P(N, n)$ becomes very small if $n \to N$ because of the smallness of the factor $p^n$ when $p \ll 1$. Hence $P(N, n)$ is indeed only appreciable when $n \ll N$.) Several approximations can then be made to reduce Eq. (1) to simpler form.
(a) Show that \((1 - p)^{N-n} \approx e^{-Np}\).

(b) Show that \(N!(N-n)! \approx N^n\).

(c) Show that Eq. (1) reduces to
\[
P(N, n) \approx \frac{(Np)^n}{n!} e^{-Np}. \tag{2}
\]
This probability distribution is called “Poisson distribution”.

(d) Show that this distribution is properly normalized in the sense that \(\sum_{n=0}^{N} P(N, n) = 1\), and compute \(\langle n \rangle\) and \(\langle (\Delta n)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2\).

(e) Summarize the difference between the Gaussian and Poisson probability distributions. Under what conditions is each of them a good approximation to the binomial distribution?

4. Radioactive decay (Reif, 1.12)

Consider \(\alpha\)-particles emitted by a radioactive source during some time interval \(t\). One can imagine this time interval to be subdivided into many small intervals of length \(\Delta t\). Since \(\alpha\) particles are emitted at random times, the probability of radioactive disintegration occurring during any such time \(\Delta t\) is completely independent of whatever disintegrations occurred previously. Furthermore, we can choose \(\Delta t\) small enough that the probability of more than one atom decaying is negligibly small.

(a) Argue that the probability \(P(n, t)\) of \(n\ \alpha\)-particles emitted in a time \(t\) is given by a Poisson distribution.

(b) Suppose that a given radioactive source emits on average 24 \(\alpha\)-particles per minute. What is the probability of obtaining \(n\) counts in a time interval of 10 seconds? Evaluate your result for all integral values of \(n\) from 0 to 8.

(c) Suppose that you monitor the source in part (b) for a year. What is the probability of obtaining at least one fluctuation of at least 8 counts in 10 seconds over the course of your experiment?