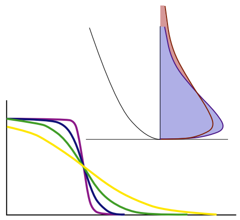
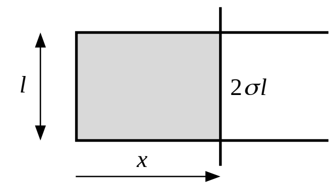


*Statistical Mechanics*

# Homework Assignment 7

Due 30 October 2002

1. **Problem 10.13 – Soap films (Reif 5.15)**




The figure illustrates a soap film (shown in gray) supported by a wire frame. Because of surface tension the film exerts a force $2\sigma l$ on the cross wire. This force is in such a direction that it tends to move this wire to decrease the area of the film. The quantity $\sigma$ is called the "surface tension" of the film and the factor 2 occurs because the film has two surfaces.

The temperature dependence of $\sigma$ is given by

$$\sigma = \sigma_0 - \alpha T$$

where $\sigma_0$ and $\alpha$ are constants independent of $T$ and $x$.

(a) Suppose that the distance $x$ (or equivalently, the total film area $2lx$) is the only external parameter of significance in the problem. Write a relation expressing the change $dE$ in mean energy of the film in terms of the thermal energy $T\,dS$ absorbed by it and the work done on it in an infinitesimal quasi-static process in which the distance $x$ is changed by an amount $dx$.

(b) Calculate the change in mean energy $\Delta E = E(x) - E(0)$ of the film when it is stretched at a constant temperature $T_0$ from a length $x = 0$ to a length $x$.

(c) Calculate the work $W(0 \to x)$ done on the film in order to stretch it at this constant temperature from a length $x = 0$ to a length $x$.

2. **Problem 10.16 – Magnetic Cooling** A paramagnetic material sample placed in an external magnetic field $\mathbf{H}$ develops a magnetic moment $\mathbf{M} = \chi\mathbf{H}$, where $\chi(T, H)$ is called the magnetic susceptibility of the material. The interaction energy between the external field and the magnetic moment of the sample adds a term to the fundamental relation of thermodynamics, giving

$$dE = T\,dS - M\,dH$$

(We will ignore the work term $-p\,dV$, since volume changes are negligible for the application we consider here.)

Imagine cooling the sample in the presence of a large magnetic field $H$, and then thermally isolating it while gradually reducing the external magnetic field $H$. If the field is reduced slowly enough, and no heat is allowed to flow into or out of the sample, the demagnetization proceeds isentropically.

(a) Show that under these circumstances

$$\left(\frac{\partial T}{\partial H}\right)_S = -\frac{TH}{C_H}\left(\frac{\partial \chi}{\partial T}\right)_H,$$

where $C_H$ is the heat capacity at constant magnetic field $H$.

(b) Show further that

$$\left(\frac{\partial C_H}{\partial H}\right)_T = TH\left(\frac{\partial^2 \chi}{\partial T^2}\right)_H.$$

These two results and a knowledge of the zero-field heat capacity $C_H(T, 0)$ allow one to compute $C_H(T, H)$ for all fields $H$ and to find $\left(\frac{\partial T}{\partial H}\right)_S$, from which one can compute the adiabatic demagnetization cooling.

3. **Problem 10.15 – A Plastic rod (Reif 5.14)** In a temperature range near absolute temperature $T$, the tension force $|\vec{\mathbf{F}}|$ of a stretched plastic rod is related to its length $L$ by the expression

$$|\vec{\mathbf{F}}| = aT^2(L - L_0)$$

where $a$ and $L_0$ are positive constants, $L_0$ being the unstretched length of the rod. When $L = L_0$, the heat capacity $C_L$ of the rod (measured at constant length) is given by the relation $C_L = bT$, where $b$ is a constant.

   (a) Write down the fundamental thermodynamic relation for this system, expressing $dS$ in terms of $dE$ and $dL$.

   (b) The entropy $S(T, L)$ of the rod is a function of $T$ and $L$. Compute $\left(\frac{\partial S}{\partial L}\right)_T$.

   (c) Knowing $S(T_0, L_0)$, find $S(T, L)$ at *any* other temperature $T$ and length $L$, within the range of applicability of the given relations for $F$ and $C_L$. (It is most convenient to calculate first the change in entropy with temperature at the length $L_0$ where the heat capacity is known.)

   (d) If one starts at $T = T_i$ and $L = L_i$ and stretches the thermally insulated rod quasi-statically until it attains the length $L_f$, what is the final temperature $T_f$? Is $T_f$ larger or smaller than $T_i$?

   (e) Calculate the heat capacity $C_L(L, T)$ of the rod when its length is $L$ instead of $L_0$.

   (f) Calculate $S(T, L)$ by writing $S(T, L) - S(T_0, L_0) = [S(T, L) - S(T_0, L)] + [S(T_0, L) - S(T_0, L_0)]$ and using the result of the previous part to compute the first term in the square brackets. Show that the final answer agrees with that found in part $(c)$.