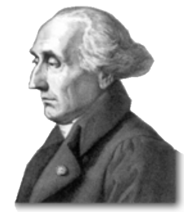
Recitation

Wednesday, 4 December 2013

Physics 111



While you are working on your projects, it is time to work practice problems in class to help you review the material of the course. There are many more, of course, in the text by Helliwell and Sahakian.

Problem 1 – Marion 8-16 A particle executes elliptical (but almost circular) motion about a force center. At some point in the orbit a *tangential* impulse is applied to the particle, changing the velocity from v to $v + \delta v$. Show that the resulting relative change in the major and minor axes of the orbit is twice the relative change in the velocity and that the axes are *increased* if $\delta v > 0$.

Problem 2 – Marion 7-12 A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

Problem 3 Find the force law for a central-force field that allows a particle to move in a logarithmic spiral orbit given by $r = ke^{\alpha\theta}$, where k and α are constants.

Problem 4 – Marion 7-20 A circular hoop is suspended in a horizontal plane by three strings, each of length l, which are attached symmetrically to the hoop and are connected to fixed points lying in a plane above the hoop. At equilibrium, each string is vertical. Show that the frequency of small rotational oscillations about the vertical through the center of the hoop is the same as that for a simple pendulum of length l.

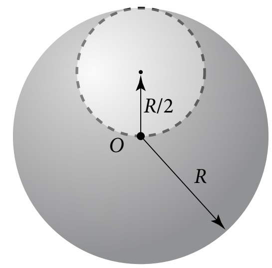
Problem 5 – Marion 11-15 If a physical pendulum has the same period of oscillation when pivoted about either of two points of unequal distances from the center of mass, show that the length of the simple pendulum with the same period is equal to the separation of the pivot points. Such a physical pendulum, called Kater's reversible pendulum, at one time provided the most accurate way (to about 1 part in 10^5) to measure the acceleration of gravity. What is the advantage of Kater's pendulum?

Problem 6 – O Holy Sphere A homogeneous sphere S has mass M and radius R. ***Note: do not use decimals in this problem. Use fractions only.***

(a) Show that the moment of inertia of the sphere about its center is $\frac{2}{5}MR^2$.

(b) What is the inertia tensor of the sphere, I_{ij}, with respect to its center?

(c) Liposuction is performed on the sphere to remove a small sphere of radius $R/2$ extending from the center of the sphere, O, to a point on its surface, as illustrated in the figure. After removing the small sphere, what is the distance a between O and the center of mass?



(d) Prove Steiner's parallel axis theorem, which is

$$J_{ij} = I_{ij} + M(r^2\delta_{ij} - r_i r_j)$$

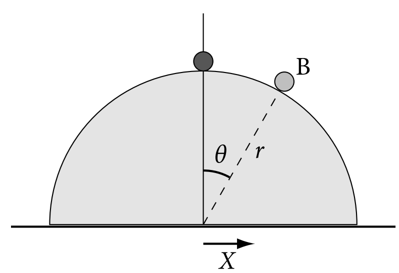
where I_{ij} is the inertia tensor with respect to the center of mass, r_i is the parallel displacement of the origin from the center of mass, M is the mass of the body, and J_{ij} is the inertia tensor with respect to the displaced origin.

(e) Compute the inertia tensor J_{ij} of the modified sphere about O, taking the position of the center of the spherical cavity with respect to O to be $\mathbf{r} = \frac{R}{2}\hat{\mathbf{z}}$. Note: if $A + B = C$, then $A = C - B$.

(f) S is placed on a smooth table and displaced slightly from its equilibrium position. It rolls without slipping. Find the period of small oscillations. Justify any approximations you make. $\left[\text{Ans: } 2\pi\sqrt{\dfrac{177R}{10g}}\right]$

Problem 7 – An Alaskan Nightmare

The onset of "winter" weather reminds me of a "popular" game played in Alaska on cold winter days when bundled-up little kids are plentiful. I'm not sure but I think Isabel Bush was telling me about it when we were killing time in Kenya. Anyway, the big kids pack together a large perfect hemisphere of snow and ice, its bottom surface perfectly smooth and plane, its upper surface smooth, slick, and spherical with radius R. The hemisphere is placed on a smooth, horizontal patch of lake ice and a slicked-up little kid (B) is positioned exactly at the top of the hemisphere.



[Okay, for those of you who are less cruel, you can imagine a bowling ball up there.] The ball/kid B (of mass m) is released from rest from the top of the hemisphere of mass M (also initially at rest). Measuring the angular position of B from vertically up with angle θ, its radial distance r, and the horizontal displacement of the hemisphere with X, as illustrated in the figure.

(a) Write down the equations of transformation for the position of B (neglect said object's spatial extent here and henceforth).

(b) Work out an expression for the kinetic energy of B.

(c) Compute the Lagrangian of the system of B and hemisphere.

(d) Using λ as the Lagrange undetermined multiplier for the radial equation, find the three Euler-Lagrange equations.

(e) Simplify the Euler-Lagrange equations under the assumption that M is effectively infinite. Show in this case that B leaves the hemisphere at $\theta_0 \approx 48.2°$ (but please leave your answer in a more mathematical form).

(f) If the mass ratio, $\mu = M/m$, is finite, will the angle at which B leaves the hemisphere be greater than, less than, or equal to θ_0? Explain your reasoning briefly.

(g) Solve numerically for the angle at which B leaves the surface if $\mu = 5$.

Problem 8 – Lagrange Points The 12 November 2010 edition of Robert Park's weekly column "What's New" discusses a problem that one had reason to hope would be remedied shortly after the Bush administration left office:

> *There are two obvious places to locate space observatories. They were identified by the great French-Italian mathematician Joseph Lagrange 237 years ago, long before anyone even imagined space observatories. The Lagrange points mark positions where the combined gravitational pull of the two large masses (Earth and Sun) provides precisely the centripetal force required to rotate a relatively small mass (the observatory) with them. There are five Lagrange points in the Earth-Sun system. The first two are the important ones. L_1 is about [distance withheld] from Earth on a line to the Sun. It is the perfect position from which to monitor the Sun in one direction, and the full illuminated Earth in the other. It is thus ideally situated to monitor changes in Earth's albedo. Americans paid more than \$100 million for an observatory at L_1, now called dscovr, the deep space climate observatory. For unexplained reasons it is sitting idle in a warehouse in Greenbelt, MD. The L_2 point is [also withheld] from Earth on a line directly away from the Sun. It is patiently waiting for the James Webb space telescope.*[1]

In this problem, you will explore the Lagrange points of the Earth-Sun system under the following simplifying assumptions:

1. The Earth's orbit is a perfect circle.
2. The influence of the Moon and of all other bodies of the solar system may be neglected.

Let m_1 be the solar mass, m_2 be the mass of the Earth, $M = m_1 + m_2$, and $\rho \equiv m_2/M$. Furthermore, call the Earth-Sun separation D, measure distances in units of D, and measure time in units of years.

(a) Explain qualitatively why it makes sense that there are two points as described by Park where a satellite of negligible mass would orbit the center of mass of the Sun-Earth system with a period of one year.

[1] At present, DSCOVR is due to launch in January, 2014, and the James Webb space telescope launch is planned for 2018.

(b) Where along the line joining the Sun and Earth should one look for a third Lagrange point? A qualitative answer will suffice.

(c) It is convenient to analyze this problem in a frame rotating once per year about the center of mass of the Sun-Earth system. This frame is not inertial, so you must include appropriate pseudo forces. We will use a rotating Cartesian coordinate system centered on the center of mass, such that the Earth is along the positive x axis and the y axis lies in the plane of the orbit. [For the lawyers among you—and we know who you are—the y axis is perpendicular to the x axis and to $\mathbf{L}$, the angular momentum of the Sun-Earth system in an inertial frame.] Show that the equations of motion of a satellite of mass $m_3 \ll m_2$ are

$$\ddot{x} - 2\Omega\dot{y} = -\frac{\partial u}{\partial x} \qquad \text{and} \qquad \ddot{y} + 2\Omega\dot{x} = -\frac{\partial u}{\partial y}$$

where Ω is the angular velocity of the rotating frame with respect to an inertial frame and u is an effective potential *per unit mass*, given by

$$u(x,y) = -\frac{\Omega^2}{2}(x^2+y^2) - G\left(\frac{m_1}{r_{31}} + \frac{m_2}{r_{32}}\right)$$

(d) Find (but don't yet solve) the equations yielding the positions (x, y) of Lagrange points. Take $M = 1$. *Hint: A particle placed at a Lagrange point remains at rest in the rotating frame. Your expressions should involve only x, y, ρ, Ω, r_{31}, r_{32}, and numerical factors. Note that a circular orbit satisfies $\Omega^2 D = GM/D^2$, so in our units with $D = 1$ and $M = 1$, $G = \Omega^2$.*

(e) There are two Lagrange points, L_4 and L_5, that do not lie on the x axis. Show that each makes an equilateral triangle with the Sun and Earth. *Hint: Look at the y equation from the previous question.*

(f) Which, if any, of the three Lagrange points along the x axis is stable with respect to small perturbations in the x direction, provided you ignore the Coriolis pseudo force?

(g) Numerically solve (using *Mathematica*) for the positions of the five Lagrange points in the (dimensionless) units we are using, assuming that $\rho = 3 \times 10^{-6}$. You may wish to use NSolve and/or FindRoot.

(h) Are the Lagrange points along the Sun-Earth axis minima, maxima, or saddle points?

(i) How far are L_1 and L_2 from Earth, in kilometers? The Earth is 1.496×10^8 km from the Sun.

(j) Extra credit: Investigate the stability of the L_4 point for small perturbations in the xy plane, when you do include the Coriolis terms. A numerical calculation using the value of ρ given above will suffice.

Problem 9 ***For this problem, use no numerical values other than those provided herein.*** The acceleration due to gravity at the surface of the Earth is $g = 9.8000\,\mathrm{m/s^2}$; the Moon takes $T = 27.300$ days to orbit the Earth. Light takes $t_c = 1.2600\,\mathrm{s}$ to travel from the surface of the Earth to the surface of the Moon along a radial line, and travels at $c = 3.0000 \times 10^8\,\mathrm{m/s}$. Assume that the Moon's mass is exactly $\frac{1}{80}$ times the Earth mass and that its radius is exactly $\frac{1}{4}R_E$. [Yes, I know these values are not quite right; deal with it! They are *exactly right* for purposes of this problem.]

Figure 1: *The International Space Station*

(a) Compute the radius of the Earth, R_E.

(b) The International Space Station (ISS) orbits the Earth at a mean altitude of $h = 340\,\mathrm{km}$. For purposes of this problem, we will assume that its orbit is circular. Compute the period of the ISS's orbit, and the velocity of the station (with respect to the Earth).

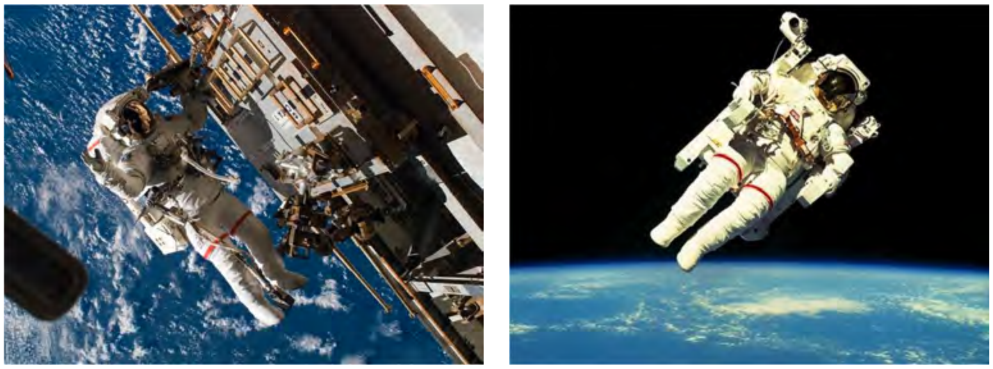


Figure 2: *Oops!*

(c) You are on a spacewalk working to assemble some backup solar panels on the ISS when something goes terribly wrong and you become untethered. By the time you struggle to correct the situation, you discover that you are 100 m from the ISS and following the ISS in the same circular orbit. Fortunately, you have still in your hand one of those fancy $50,000 gold-plated wrenches NASA provides.

Its mass is $m = 0.5\,\text{kg}$, but yours (including your extensive space suit and breathing apparatus) is $100\,\text{kg}$. You also have a calculator and a 2-hour oxygen supply. To get back to the ISS, you decide to throw the wrench. Should you throw the wrench towards the station or away from it to get back in the least time? Explain carefully.

(d) Assuming that you can throw the wrench at up to $10\,\text{m/s}$ with respect to you and that you throw it in the direction you picked to get back most expeditiously, how fast should you throw it to arrive at the station in 91 minutes?