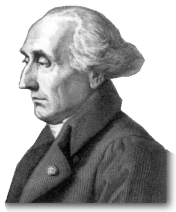
Lagrangian Mechanics I

Wednesday, 11 September 2013

How Lagrange revolutionized Newtonian mechanics

Physics 111



Newton based his dynamical theory on the notion of force, for which he gave a somewhat incomplete definition, but which he made plausible by appealing to the common experience of pushes and pulls to which the human body may be subject. In the century following Newton's publication of the *Philosophiæ Naturalis Principia Mathematica* (mathematical principles of natural philosophy), mathematicians sought to clarify and generalize the foundation he laid. The most important developments before Einstein were due to Leonhard Euler (1707–1783), Joseph-Louis Lagrange (1736–1813), and William Rowan Hamilton (1805–1865). Euler and Lagrange developed the general formulation of the calculus of variations, Lagrange applied the method of virtual velocities developed initially by Johann Bernoulli (1667–1748), along with generalized coordinates, to produce a powerful new approach to solving dynamics problems. This method today is called Lagrangian mechanics and remains the method of choice for solving a variety of mechanics problems involving conservative interactions. Hamilton provided a third approach, derived from Lagrange's, which proved fundamental to the foundation of statistical mechanics and quantum mechanics. We will investigate Hamilton's approach after making a careful study of Lagrange's.

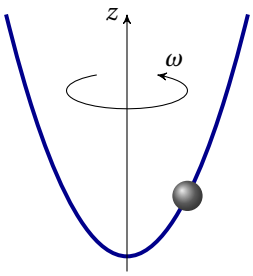
One way to develop Lagrangian mechanics is from Hamilton's principle, which states

> *Of all the possible paths along which a dynamical system may move from one point to another in configuration space within a specified time interval, the actual path followed is that which minimizes the time integral of the difference between the kinetic and potential energy of the system.*

This approach has the advantage of simplicity and power, and is roughly the approach used by Helliwell and Sahakian, who are inspired by the motion of a (relativistic) particle in a uniform gravitational field at the conclusion of Chapter 3. The disadvantage of this approach is that it seems quite disconnected from the Newtonian roots from which it sprang. Furthermore, the Irish mathematician and physicist William Rowan Hamilton wasn't born until thirteen years after Lagrange published his great work, *Mécanique analytique*, in 1788. To supplement the text, we will develop here Lagrange's approach following a path that more closely approximates history. Our concession to modernity will be to use the terms **kinetic energy**, **potential energy**, and **work**, rather than those Lagrange and his contemporaries actually employed. We will develop Lagrange's approach by studying a relatively simple dynamical system, after which we will generalize.

1. Example System

A small bead of mass m is threaded on a wire bent in the shape of a parabola of equation $z = \alpha x^2$, where z is along the vertical and α is a constant. The wire is mounted to a motor and made to rotate at the constant angular speed ω about the z axis. We seek the equation(s) of motion of the bead.



In the Newtonian approach, we prepare an isolation diagram showing all the forces applied to the bead (gravity and the normal force supplied by the wire), and set $\mathbf{F} = m\mathbf{a}$. Our first step in developing the Lagrangian approach is to make the very mundane observation that

$$\sum \mathbf{F}^{\text{ext}} - m\ddot{\mathbf{r}} = 0 \tag{1}$$

That is, the sum of all forces, including the "**inertial force**" $-m\ddot{\mathbf{r}}$, is zero, just as in static equilibrium. This is not particularly revolutionary, as far as it goes, but is sometimes employed as a way of avoiding sign errors by including "inertial forces" on free-body diagrams.

The insight that goes under the name **d'Alembert's principle**, after Jean le Rond d'Alembert (1717–1783), but is based as well on the developments of Johann Bernoulli and Leonhard Euler, is to imagine the work done in an infinitesimal, instantaneous "virtual" displacement of the bead $\delta\mathbf{r}$ in a direction consistent with the constraints on the bead's motion. In this case, the bead's motion through three-dimensional space is really confined to the one-dimensional parabolic curve of the rotating wire. Such a constraint, which reduces the number of coordinates (also called **degrees of freedom**) necessary to specify the position of the particle, is called **holonomic**. Lagrange's approach significantly simplifies dynamics problems that involve holonomic constraints.

A **virtual displacement** is an instantaneous displacement of a particle that respects all constraints. We will use a δ to represent virtual displacements.

A **nonholonomic** constraint limits the range of motion of a particle or system without reducing the number of degrees of freedom. For example, a particle that moves inside a spherical shell, such that $r < a$. In such a case, the approach we develop here has no advantage over the more familiar Newtonian approach.

When we include the inertial forces, the sum of all forces vanishes by Eq. (1), so no work is done in the virtual displacement,

$$\left(\sum \mathbf{F}^{\text{con}} + \sum \mathbf{F}^{\text{app}} - m\ddot{\mathbf{r}}\right) \cdot \delta\mathbf{r} = 0 \tag{2}$$

where we have separated the external forces on m into constraint forces and (all the other) applied forces. At this point we note that the virtual displacement is along the wire and therefore perpendicular to the forces of constraint. Hence, $\sum \mathbf{F}^{\text{con}} \cdot \delta\mathbf{r} = 0$, which implies **d'Alembert's principle**,

$$\left(\sum \mathbf{F}^{\text{app}} - m\ddot{\mathbf{r}}\right) \cdot \delta\mathbf{r} = 0 \tag{3}$$

a nontrivial simplification of the problem. In our example, the only applied force is gravity, which in the coordinate system indicated in the figure takes the form $\mathbf{F} = -mg\hat{\mathbf{z}}$.

The acceleration in cylindrical coordinates is somewhat involved. First, the position vector is $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$. Differentiating once with respect to time gives

The position vector has no dependence on $\hat{\boldsymbol{\phi}}$, since to get from the origin to any point we can walk up or down the z axis until we are even with the point, then walk radially outward to it.

$$\dot{\mathbf{r}} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}} \tag{4}$$

where the dependence on ϕ arises because the direction of $\hat{\boldsymbol{\rho}}$ changes if the value of ϕ changes.[1] Differentiating again and collecting terms gives

$$\ddot{\mathbf{r}} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{z}} \tag{5}$$

where $\dot{\phi} = \omega = \text{constant}$.[2] The virtual displacement $\delta\mathbf{r}$ must be along the wire ($z = \alpha\rho^2$), and therefore have slope

$$\frac{dz}{d\rho} = 2\alpha\rho \qquad \Longrightarrow \qquad \delta z = 2\alpha\rho\delta\rho$$

Substituting into Eq. (3), we have

$$\left[-mg\hat{\mathbf{z}} - m\left(\ddot{\rho} - \rho\omega^2\right)\hat{\boldsymbol{\rho}} - m\ddot{z}\hat{\mathbf{z}}\right] \cdot \left(\hat{\boldsymbol{\rho}} + 2\alpha\rho\hat{\mathbf{z}}\right)\delta\rho = 0 \tag{6}$$

or

$$-(g + \ddot{z})2\alpha\rho - \ddot{\rho} + \rho\omega^2 = 0 \tag{7}$$

At this point, we need again to invoke the constraint equation $z = \alpha\rho^2$ to obtain

$$\begin{aligned}\dot{z} &= 2\alpha\rho\dot{\rho} \\ \ddot{z} &= 2\alpha\dot{\rho}^2 + 2\alpha\rho\ddot{\rho}\end{aligned}$$

from which we find the equation of motion,

$$\begin{aligned}2\alpha\rho(g + 2\alpha\dot{\rho}^2 + 2\alpha\rho\ddot{\rho}) + \ddot{\rho} - \rho\omega^2 &= 0 \\ \ddot{\rho}(1 + 4\alpha^2\rho^2) + 4\alpha^2\rho\dot{\rho}^2 + \rho(2\alpha g - \omega^2) &= 0\end{aligned} \tag{8}$$

You may derive this same equation through more involved means using the Newtonian approach. The virtual-displacement trick quickly eliminated the constraint forces, allowing us to proceed more directly to the equations of motion. On the other hand, we still needed the somewhat complicated expression for the acceleration in cylindrical coordinates, Eq. (5), as well as vector components in taking the dot product.

2. Generalized Coordinates

Lagrange's key insight was to generalize the method of virtual work to use an arbitrary set of **generalized coordinates**, which are a set of real numbers (possibly with units) that suffice to specify the position of the particle uniquely. That is, we assume that we have a set of coordinates q_i such that we may express the particle position as

$$\mathbf{r} = \mathbf{r}(q_i, t) \tag{9}$$

The equation of transformation expresses the cartesian coordinates of the particle in terms of the chosen generalized coordinates q_i.

[1] You can show that $\frac{d\hat{\boldsymbol{\rho}}}{d\phi} = \hat{\boldsymbol{\phi}}$ directly by expressing both $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ in cartesian coordinates and differentiating the former with respect to ϕ.

[2] The course text uses r for the distance from the z axis in cylindrical coordinates. To avoid confusion between $\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$ and a unit vector in the radial direction, I will use ρ for the radial distance in the following.

which is called the **equation of transformation**. The coordinates q_i may be any convenient values, such as angles, distances, or combinations thereof. In many circumstances cartesian coordinates are just not well suited to the geometry of a problem and it will be well worth our while to learn how to generalize to curvilinear coordinates. In this case, we will be using cylindrical coordinates, (ρ, ϕ, z), which lack explicit time dependence, but to keep the discussion general, we will seek to redo the derivation of the previous section using the coordinates q_i.

The particle's velocity may be expressed by the chain rule of vector calculus:

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \sum_i \frac{\partial \mathbf{r}}{\partial q_i}\dot{q}_i + \frac{\partial \mathbf{r}}{\partial t} \tag{10}$$

where the $\dot{q}_i \equiv \dfrac{dq_i}{dt}$ are called the **generalized velocities** of the particle.

If we may assume that $\mathbf{r}$ is a function of the coordinates q_i and the time t, *but not of the generalized velocities* $\dot{q}_i$, then the velocity $\mathbf{v}$ is a function of both the generalized coordinates and the generalized velocities, but the dependence on the $\dot{q}_i$ is simply linear. That is, taking a partial derivative with respect to either q_i or t of a function of only (q_i, t) cannot introduce dependence on $\dot{q}_i$. Therefore, when we take a partial derivative of Eq. (10) with respect to $\dot{q}_i$ we get

$$\frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_i} = \frac{\partial \mathbf{r}}{\partial q_i} \tag{11}$$

as though we can just cancel the dots. This only works because the equations of transformation, $\mathbf{r}(q_j, t)$, have no *explicit* dependence on the coordinate velocities $\dot{q}_j$. We will need this relation shortly.

This result requires that the equation of transformation, $\mathbf{r} = \mathbf{r}(q_i, t)$, has no *explicit* dependence on the $\dot{q}_i$.

Question 1 Suppose, instead, that the particle position is a function of both the generalized coordinates and the generalized velocities, $\mathbf{r} = \mathbf{r}(q_i, \dot{q}_i, t)$. Explain why Eq. (11) no longer holds.

2.1 Virtual work in generalized coordinates

We now seek to express Eq. (3) using generalized coordinates, since Prof. Lagrange promises us that this will simplify our life. In particular, we will attack the term $m\ddot{\mathbf{r}} \cdot \delta\mathbf{r}$. Again using the chain rule, we have

$$\delta\mathbf{r} = \sum_i \frac{\partial \mathbf{r}}{\partial q_i}\delta q_i \tag{12}$$

where there is no term $\frac{\partial \mathbf{r}}{\partial t}$ because the virtual displacement happens instantaneously. Therefore,

$$m\ddot{\mathbf{r}} \cdot \delta\mathbf{r} = \sum_i m\ddot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i}\delta q_i = m\ddot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i}\delta q_i \quad \text{(summation convention)} \tag{13}$$

Einstein grew tired of writing summation signs for product terms in which a single index was used twice, as in Eq. (12), so he invented a notational convention: repeated indices in a product are summed over unless there is a warning not to.

With malice of forethought, consider now the derivative

$$\frac{d}{dt}\left(\dot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i}\right) = \ddot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i} + \dot{\mathbf{r}} \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right) \tag{14}$$

The second term on the right-hand side is a second derivative: the position vector is differentiated with respect to both the generalized coordinate q_i and the time. Interchanging the order of derivatives, this term becomes $\dot{\mathbf{r}} \cdot \partial \dot{\mathbf{r}} / \partial q_i$. Meanwhile, the first term on the right-hand side of Eq. (14) is just the term we need. Isolating this term gives

$$\ddot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i} = \frac{d}{dt}\left(\dot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i}\right) - \dot{\mathbf{r}} \cdot \frac{\partial \dot{\mathbf{r}}}{\partial q_i} \tag{15}$$

The final term may be simplified by noting that

$$\dot{\mathbf{r}} \cdot \frac{\partial \dot{\mathbf{r}}}{\partial q_i} = \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v}\right)$$

and we can pull a similar trick with the first term on the right-hand side of Eq. (15) by invoking Eq. (11) to replace coordinates with velocities. Therefore,

$$\ddot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i} = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_i}\left(\frac{1}{2}v^2\right)\right] - \frac{\partial}{\partial q_i}\left(\frac{1}{2}v^2\right) \tag{16}$$

Multiplying through by mass m and calling $T = mv^2/2$ the **kinetic energy** of the particle, we have

$$m\ddot{\mathbf{r}} \cdot \frac{\partial \mathbf{r}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} \tag{17}$$

or

$$m\ddot{\mathbf{r}} \cdot \delta\mathbf{r} = \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right]\delta q_i \tag{18}$$

where we are implicitly summing over i.

We now need to express the other term in Eq. (3) in terms of the generalized coordinates,

$$\mathbf{F}^{\text{app}} \cdot \delta\mathbf{r} = \sum_i \mathbf{F}^{\text{app}} \cdot \frac{\partial \mathbf{r}}{\partial q_i}\delta q_i = \sum_i \mathscr{F}_i\, \delta q_i \tag{19}$$

where the $\mathscr{F}_i$ are the components of the **generalized force**, defined by

$$\mathscr{F}_i = \mathbf{F}^{\text{app}} \cdot \frac{\partial \mathbf{r}}{\partial q_i} \tag{20}$$

Hence, Eq. (3) expressed in generalized coordinates becomes

$$\sum_i \left\{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right] - \mathscr{F}_i\right\}\delta q_i = 0 \tag{21}$$

Since each of the virtual displacements δq_i is independent of the others,[3] this implies that the term in braces must vanish for each generalized coordinate:

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \mathscr{F}_i = 0} \tag{22}$$

Question 2 What are the physical dimensions of the generalized force $\mathscr{F}_i$ defined by Eq. (20)?

[3] Sometimes it will be more convenient to use redundant coordinates, such as ρ and z in the example, in which case we will have to ensure that their variations are coordinated. This has the additional benefit of yielding the constraint forces, should they be of interest. For the moment, however, we will assume that the q_i constitute a minimal set necessary to specify the position of the particle, which in this case is the single coordinate ρ.

2.2 Conservative Forces

A further simplification is possible if the applied forces are conservative, and can thus be derived from a scalar potential. Recall that the curl of a gradient vanishes, since

$$\nabla\times(\nabla U) = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial U}{\partial x} & \frac{\partial U}{\partial y} & \frac{\partial U}{\partial z} \end{vmatrix}$$
$$= \hat{\mathbf{x}}\left(\frac{\partial^2 U}{\partial y\partial z} - \frac{\partial^2 U}{\partial z\partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 U}{\partial z\partial x} - \frac{\partial^2 U}{\partial x\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 U}{\partial x\partial y} - \frac{\partial^2 U}{\partial y\partial x}\right) = 0$$

by the equality of mixed partial derivatives. If a force satisfies $\nabla\times\mathbf{F}=0$, then there is a scalar function U such that $\mathbf{F}=-\nabla U$. Then

Conservative forces satisfy $\nabla\times\mathbf{F}=0$.

$$\mathscr{F}_i = \mathbf{F}^{\rm app}\cdot\frac{\partial\mathbf{r}}{\partial q_i} = -\nabla U\cdot\frac{\partial\mathbf{r}}{\partial q_i} = -\left(\frac{\partial U}{\partial x}\frac{\partial x}{\partial q_i} + \frac{\partial U}{\partial y}\frac{\partial y}{\partial q_i} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial q_i}\right) = -\frac{\partial U}{\partial q_i} \tag{23}$$

Hence,

$$\frac{\partial(T-U)}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_i}\right) = 0 \tag{24}$$

It is annoying that the first term differentiates $T-U$ while the second only differentiates T. Of course, if U doesn't have any *explicit* dependence on the generalized velocities $\dot{q}_i$, then adding $\frac{\partial U}{\partial\dot{q}_i}$ causes no harm. But that is precisely what we are assuming: that the applied force depends only on positions, not generalized velocities, $\frac{\partial U}{\partial\dot{q}_i}=0$. We thus define the **lagrangian**,

$$L(q_i,\dot{q}_i,t) = T-U \tag{25}$$

to yield (finally) **Lagrange's equation**s:

The master equation of Lagrangian mechanics for conservative systems. Commit it to memory!

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0} \tag{26}$$

If some of the applied forces may be derived from scalar potentials, but others cannot, then we can simply return to Eq. (22) and separate the applied forces into conservative and nonconservative. The conservative forces are treated as above and are included in the lagrangian, while the nonconservative forces remain as in Eq. (22):

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_i}\right) + \mathscr{F}_i^{\rm noncons} = 0 \tag{27}$$

In practice, however, the Lagrangian formulation is most beneficial when any nonconservative forces may be neglected.

2.3 Cyclic Coordinates

We have called the q_i *generalized coordinates*, and their time derivatives *generalized velocities*. While we are generalizing, we may as well define

$$p_j \equiv \frac{\partial L}{\partial\dot{q}_j} \tag{28}$$

the **generalized momentum** corresponding to generalized coordinate q_j. You may readily confirm that for a "normal" cartesian coordinate, the generalized momentum is identical to the regular momentum component you are used to.

The generalized momentum defined by $\partial L/\partial \dot{q}_j$ is fundamentally important in the Hamiltonian formulation of mechanics, and by extension, in quantum mechanics.

A cursory examination of Eq. (26) shows that something special happens when the Lagrangian has no explicit dependence on a particular generalized coordinate. We say that the lagrangian is **cyclic** in such a coordinate. In that case, the total time derivative of the generalized momentum corresponding to the cyclic coordinate vanishes: the generalized momentum is a constant of the motion, a conserved quantity, and there's really no reason to take that derivative. Instead, just note that

Such coordinates are also called "ignorable."

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \text{constant} \tag{29}$$

is a first integral of the motion and be happy that you only need to do one more integral to obtain the position as a function of time.

3. Example

We have loaded the elephant gun; let's shoot the bead-on-the-parabolic-wire problem. The potential energy is simple: $U = mgz$. To find the kinetic energy in cylindrical coordinates, start with the velocity,

$$\dot{\mathbf{r}} = \mathbf{v} = \dot{\rho}\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\hat{\boldsymbol{\phi}} + \dot{z}\hat{\mathbf{z}} \tag{30}$$

so that

$$T = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right) \tag{31}$$

and the lagrangian is

$$L = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\omega^2 + \dot{z}^2\right) - mgz \tag{32}$$

Using the equation of the parabola, we can eliminate z to get

$$L = \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\omega^2 + 4\alpha^2\rho^2\dot{\rho}^2\right) - mg\alpha\rho^2 \tag{33}$$

Note that we must do this because the particle has only a single **degree of freedom**. That is, only one generalized coordinate is required to uniquely specify the position of the particle. Later we will see how to manage constraints without reducing the number of coordinates. Now we use Lagrange's equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = \frac{d}{dt}\left(m\dot{\rho} + 4m\alpha^2\rho^2\dot{\rho}\right) - m\omega^2\rho - 4m\alpha^2\dot{\rho}^2\rho + 2mg\alpha\rho = 0 \tag{34}$$

Carrying out the differentiation, and dividing by m, we obtain the equation of motion for the coordinate ρ:

$$\begin{aligned}\ddot{\rho} + 4\alpha^2\rho^2\ddot{\rho} + 8\alpha^2\rho\dot{\rho}^2 - \rho(\omega^2 + 4\alpha^2\dot{\rho}^2) + 2g\alpha\rho = 0 \\ \ddot{\rho}(1 + 4\alpha^2\rho^2) + 4\alpha^2\rho\dot{\rho}^2 + \rho(2\alpha g - \omega^2) = 0\end{aligned} \tag{35}$$

which is the same equation we had before. It may seem like this was an awful lot of work to get just what we had already, but most of that work was spent deriving Eq. (26). In curvilinear coordinates it is invariably simpler to compute the velocity than the acceleration, so the lagrangian is more straightforward to obtain than the Newtonian equations. Furthermore, once L has been computed, Eq. (26) provides a very straightforward path to the equation(s) of motion.

4. General Procedure

Let us take a moment to summarize what we have seen so far. The general approach to solving a dynamics problem using the Lagrangian formalism is as follows:

1. Select a minimal set of generalized coordinates q_i, one for each degree of freedom.
2. Express the square of the velocity in terms of the generalized coordinates, $v^2(q_i, \dot{q}_i, t)$, and similarly the kinetic energy $T = \frac{1}{2}mv^2$.
3. Express the potential energy, U, of the particle in terms of the generalized coordinates.
4. Form the lagrangian, $L(q_i, \dot{q}_i, t) = T - U$.
5. Note any cyclic coordinates, which imply conserved generalized momenta.
6. Obtain the equation of motion for each noncyclic generalized coordinate from the lagrangian using the Lagrange equation $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \dfrac{\partial L}{\partial q_i} = 0$.

Remember that the (total) time derivative in this equation differentiates every variable (q_i, $\dot{q}_i$, and t) that appears in $\partial L/\partial \dot{q}_i$.

Although it may seem complicated at first blush, in many respects Lagrange's method is much simpler than Newton's. Energy is a scalar, which makes it easier to work with than forces and accelerations. It eliminates forces of constraint entirely. Once you have settled on a set of generalized coordinates, your path to obtain the equations of motion lies clear before you, although it may require a page or two of careful algebra.