Chapter 4

Lagrangian mechanics

Motivated by discussions of the variational principle in the previous chapter, together with the insights of special relativity and the principle of equivalence in finding the motions of free particles and particles in uniform gravitational fields, we seek now a variational principle for the motion of nonrelativistic particles subject to *arbitrary* forces. This will lead us to introduce **Hamilton's principle** and the **Lagrangian** to describe physical systems in mechanics, both for single particles and systems of particles. These concepts together are so elegant that we are encouraged to place them at the very heart of classical mechanics. We are further encouraged to do so in the following chapter, the capstone chapter to Part I of the book, where we show how they naturally emerge as we take the classical limit of the vastly more comprehensive theory of quantum mechanics.

4.1 The Lagrangian in Cartesian coordinates

At the end of Chapter 3 we reached the intriguing conclusion that the correct equations of motion for a nonrelativistic particle of mass m in a uniform gravitational field can be found by making stationary the functional

$$I \to \int dt \left(\frac{1}{2} m\, v^2 - U\right) = \int dt\, (T - U), \tag{4.1}$$

where

$$T \equiv \frac{1}{2} m\, v^2 \tag{4.2}$$

is the particle's kinetic energy and

$$U = mgy \tag{4.3}$$

is its gravitational potential energy. It was the *difference* between the kinetic and gravitational potential energy that was needed in the integrand.

Now suppose that a particle is subject to an *arbitrary* conservative force for which a potential energy U can be defined. Does the form

$$I \to \int dt\, \left(\frac{1}{2}m\, v^2 - U\right) = \int dt\, (T - U) \tag{4.4}$$

still work? Do we still get the correct $\boldsymbol{F} = m\boldsymbol{a}$ equations of motion?

Let us do a quick check using Cartesian coordinates. Note that if $U = U(x, y, z)$ and $T = T(\dot{x}, \dot{y}, \dot{z})$, then the integrand in the variational problem, which we now denote by the letter L, is

$$L(x, y, z, \dot{x}, \dot{y}, \dot{z}) \equiv T(\dot{x}, \dot{y}, \dot{z}) - U(x, y, z) = \frac{1}{2}mv^2 - U(x, y, z), \tag{4.5}$$

where $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$. Writing out the three associated Euler equations, we get the differential equations of motion

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0, \qquad \frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = 0, \qquad \frac{\partial L}{\partial z} - \frac{d}{dt}\frac{\partial L}{\partial \dot{z}} = 0, \tag{4.6}$$

where

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F_x \quad \text{and} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}m\dot{x} = m\ddot{x}, \quad \text{so} \quad F_x = m\ddot{x}, \tag{4.7}$$

with similar results in the y and z directions. That is, we have derived the three components of $\boldsymbol{F} = m\boldsymbol{a}$,

$$F_x = m\ddot{x}, \qquad F_y = m\ddot{y}, \qquad F_z = m\ddot{z} \Rightarrow -\boldsymbol{\nabla}U = m\boldsymbol{a}\ . \tag{4.8}$$

The quantity

$$L = T - U \tag{4.9}$$

is called the **Lagrangian** of the particle. As we have seen, using the Lagrangian as the integrand in the variational problem gives us the correct equations of motion, at least in Cartesian coordinates, for *any* conservative force!

4.2 Hamilton's principle

We now have an interesting proposal at hand: reformulate the equations of motion of nonrelativistic mechanics, $\boldsymbol{F} = d\boldsymbol{p}/dt$, in terms of a variational principle making stationary a certain functional. This has two benefits:

(1) It is an interesting and intuitive idea to think of dynamics as arising from making a certain physical quantity stationary; we will appreciate some of these aspects in due time, especially when we get to the chapter on the connections between classical and quantum mechanics;

(2) This reformulation provides powerful computational tools that can allow one to solve complex mechanics problems with greater ease. The formalism also lends itself more transparently to implementations in computer algorithms.

The Lagrange technique makes brilliant use of what are called **generalized coordinates**, particularly when the particle or particles are subject to one or more **constraints**. Suppose that a particular particle is free to move in all three dimensions, so three coordinates are needed to specify its position. The coordinates might be Cartesian (x, y, z), cylindrical (r, φ, z), spherical (r, θ, φ), as illustrated in Figure 4.1, or they might be any other complete set of three (not necessarily orthogonal) coordinates.[1]

A different particle may be less free: it might be constrained to move on a tabletop, or along a wire, or within the confines of a closed box, for example. Sometimes the presence of a constraint means that fewer than three coordinates are required to specify the position of the particle. So if the particle is restricted to slide on the surface of a table, for example, only two coordinates are needed. Or if the particle is a bead sliding along a frictionless wire, only one coordinate is needed, say the distance of the bead from a given point on the wire. On the other hand, if the particle is confined to move within a closed three-dimensional box, the constraint does *not* reduce the number of coordinates required: we still need three coordinates to specify

[1]Note that in spherical coordinates the radius r is the distance from the origin, while in cylindrical coordinates r is the distance from the vertical (z) axis. Because these r's refer to different distances, some people use ρ instead of r in cylindrical coordinates to distinguish it from the r in spherical coordinates. However, retaining the symbol r in cylindrical coordinates has the great advantage that on any $z =$ constant plane the coordinates (r, φ) automatically become a good choice for conventional planar polar coordinates.

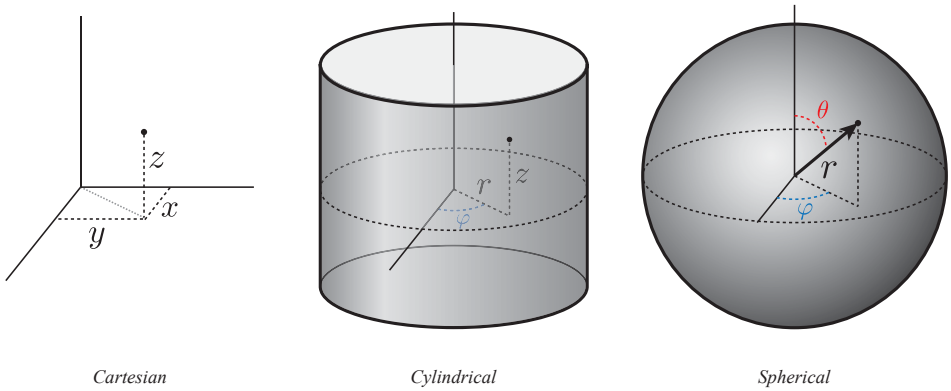

FIGURE 4.1 : Cartesian, cylindrical, and spherical coordinates

the position of the particle inside the box.

A constraint that reduces the number of coordinates needed to specify the position of a particle is called a **holonomic** constraint. The requirement that a particle move anywhere on a tabletop is a holonomic constraint, for example, because the minimum set of required coordinates is lowered from three to two, from (say) (x, y, z) to (x, y). The requirement that a bead move on a wire in the shape of a helix is a holonomic constraint, because the minimum set of required coordinates is lowered from three to one, from (say) cylindrical coordinates (r, φ, z) to just z. The requirement that a particle remain within a closed box is **nonholonomic**, because a requirement that $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$, $z_1 \leq z \leq z_2$ does not reduce the number of coordinates required to locate the particle.

For an unconstrained particle, three coordinates are needed; or if there is a holonomic constraint the number of coordinates is reduced to two or one. We call a minimal set of required coordinates **generalized coordinates** and denote them by q_k, where $k = 1, 2, 3$ for a single particle (or $k = 1, 2$, or just $k = 1$, for a constrained particle). For each generalized coordinate there is a generalized velocity $\dot{q}_k = dq_k/dt$. Note that a generalized velocity does not necessarily have the dimensions of length/time, just as a generalized coordinate does not necessarily have the dimensions of *length*. For example, the polar angle θ in spherical coordinates is dimensionless, and its generalized velocity $\dot{\theta}$ has dimensions of inverse time.

Having chosen a set of generalized coordinates q_k for a particle, the integrand L in the variational problem, where L is called the **Lagrangian**, can be written[2]

$$L = L(t, q_1, q_2, .., \dot{q}_1, \dot{q}_2, ...) = L(t, q_k, \dot{q}_i) \tag{4.10}$$

in terms of the generalized coordinates, generalized velocities, and the time.

We can now present a more formal statement of the Lagrangian approach to finding the differential equations of motion of a system.

Given a mechanical system described through N dynamical generalized coordinates labeled $q_k(t)$, with $k = 1, 2, \ldots, N$, we define its **action** $S[q_k(t)]$ as the functional of the time integral over the **Lagrangian** $L(t, q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...)$, from a starting time t_a to an ending time t_b,

$$S[q_k(t)] = \int_{t_a}^{t_b} dt\, L(t, q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...) \equiv \int_{t_a}^{t_b} dt\, L(t, q_k, \dot{q}_k)\ . \tag{4.11}$$

It is understood that the particle begins at some definite position $(q_1, q_2, ...)_a$ at time t_a and ends at some definite position $(q_1, q_2, ...)_b$ at time t_b. We then propose that, for trajectories $q_k(t)$ where S is *stationary* — *i.e.*, , when

$$\delta S = \delta \int_{t_a}^{t_b} L(t, q_k, \dot{q}_k)\ dt = 0 \tag{4.12}$$

the $q_k(t)$'s satisfy the equations of motion for the system with the prescribed boundary conditions at t_a and t_b. This proposal was first enunciated by the Irish mathematician and physicist William Rowan Hamilton (1805 – 1865), and is called **Hamilton's principle.**[3] From Hamilton's principle and our discussion of the previous chapter, we get the N **Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0. \qquad (k = 1, 2, \ldots, N) \tag{4.13}$$

[2]Here, we are assuming that the Lagrangian does not involve dependence on higher derivatives of q_k, such as $\ddot{q}_k$. It can be shown that such terms lead to differential equations of the third or higher orders in time (See Problems section). Our goal is to reproduce traditional Newtonian and relativistic mechanics involving differential equations that are no higher than second order.

[3]It is also sometimes called the *Principle of Least Action* or the *Principle of Stationary Action*. This can be confusing, however, because there is an older principle called the "principle of least action" that is quite different.

These then have to be the equations of motion of the system if Hamilton's principle is correct. Note that we *need* to use generalized coordinates since the variational principle assumes that the perturbed variables in the functional are *independent*.

Consider a general physical system involving only conservative forces and a number of particles — constrained or otherwise. We propose that we can describe the dynamics of this system fully through Hamilton's principle, using the Lagrangian $L = T - U$, the difference between the total kinetic energy and the total potential energy of the system — written in generalized coordinates. For a single particle under the influence of a conservative force, and described with Cartesian coordinates, we have already shown that this is indeed possible. The question is then whether we can extend this new technology to more general situations with several particles, constraints, and described with arbitrary coordinate systems. We will show this step by step, looking at explicit examples and generalizing from there. There are three main issues we would need to tackle in this process:

1. Does changing the coordinate system in which we express the kinetic and potential energies generate any obstacles to the formalism? The answer to this is "*no*", since the functional we extremize – which is the action – is a **scalar quantity**: its value does not change under coordinate transformations $q_k \to q'_k$

$$S = \int dt\, L(t, q_k, \dot{q}_k) = \int dt\, L(t, q'_k, \dot{q}'_k)\ . \tag{4.14}$$

   The coordinate change simply relabels the stationary path of the functional; that is, the path at the extremum transforms as $q_k^{sol}(t) \to {q'_k}^{sol}(t)$ where ${q'_k}^{sol}(t)$ is the stationary path of S expressed in the new coordinates. Hence, we can safely perform coordinate transformations as long as we always write the Lagrangian as kinetic minus potential energy in our preferred coordinate system.

2. Constraints provide for relations between the variables describing a mechanical system, and hence reduce their number to a minimal set of *generalized coordinates*. This was the premise of the variational principle: the generalized coordinates must be *independent*. Hence, no constraint would interfere with the variational principle as long as we express $L = T - U$ in terms of the generalized coordinates. But

constraints on the coordinates are due to forces in the system that restrict the dynamics. For example, the normal force pushes upward to make sure a block stays on the floor; likewise, the tension force in a rope constrains the motion of a bob pendulum. Can we be certain that these forces should not be included in the potential energy U that appears in L? To ensure that this is the case, we need to ascertain that such **constraint forces** do no work, and hence do not have any net energetic contribution to U. This is not always easy to see. We will demonstrate the mechanism at work through examples, and then identify the general strategy.

3. Finally, should we expect any obstacles to the formalism when we have more than one particle? Do we simply add the kinetic and potential energies of all the particles? With two or more particles, shouldn't we worry about Newton's third law? We will see soon that the Lagrangian formalism incorporates Newton's third law and indeed can handle many-particle systems very well.

The punchline of all this is that, for arbitrarily complicated systems with many constraints and involving many particles interacting with **conservative** forces, the Lagrangian formalism works, and is very powerful. Newton's second law follows from Hamilton's principle, and the third law arises, as we will see, from symmetries of the Lagrangian. How about the first law? That is indeed an important potential pitfall: *one should always write the kinetic energy and potential energy in $L = T - U$ as seen from the perspective of an inertial observer.* This is because our contact with mechanics is through the reproduction of Newton's second law, $\boldsymbol{F} = m\boldsymbol{a}$ — which is valid only in an inertial frame. With this in mind, we now have a proposal that reformulates Newtonian mechanics through a powerful new formalism.

It is important to emphasize that the Lagrangian formalism does *not* introduce new physics. It is a mathematical reformulation of good old mechanics, non-relativistic and relativistic. What it does is give us powerful new technical tools to tackle problems with greater ease and less work, a deep insight into the laws of physics and how Nature ticks, and how the classical world is linked to the quantum realm.

EXAMPLE 4-1: A simple pendulum

A small plumb bob of mass m is free to swing back and forth in a vertical x-z plane at the end of a string of length R. The position of the bob can be specified uniquely by its angle θ measured up from its equilibrium position at the bottom, so we choose θ as the generalized coordinate. The bob's kinetic energy is

$$T = \frac{1}{2} m\, v^2 = \frac{1}{2} m\, (\dot{x}^2 + \dot{z}^2) = \frac{1}{2} m\, (\dot{r}^2 + r^2 \dot{\theta}^2) = \frac{1}{2} m\, (R^2 \dot{\theta}^2)\ . \tag{4.15}$$

Here, we switched to polar coordinates, and implemented the constraint equations $\dot{r} = 0$ and $r = R$. Its potential energy is $U = mgh = mgR(1 - \cos\theta)$, measuring the bob's height h up from its lowest point. The Lagrangian of the bob is therefore

$$L = T - U = \frac{1}{2} mR^2 \dot{\theta}^2 - mgR(1 - \cos\theta). \tag{4.16}$$

The constraint reduces the dynamics from two planar coordinates to only one, the single degree of freedom that is the angle θ. The single Euler equation in this case is

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgR\sin\theta - \frac{d}{dt}\left(mR^2\dot{\theta}\right) = 0, \tag{4.17}$$

equivalent to the well-known "pendulum equation"

$$\ddot{\theta} + (g/R)\sin\theta = 0. \tag{4.18}$$

Note that equation (4.17)) (or (4.18)) is equivalent to $\tau = I\ddot{\theta}$, where the torque $\tau = -mgR\sin\theta$ is taken about the point of suspension (negative because it is opposite to the direction of increasing θ), and the moment of inertia of the bob is $I = m\,R^2$.

We had two twists in this problem. First, we switched from Cartesian to polar coordinates. But we know that this is not a problem for the Lagrangian formalism since the action is a scalar quantity. Second, we implemented a constraint $r = R$, implying $\dot{r} = 0$. This constraint is responsible for holding the bob at fixed distance from the pivot and hence is due to the **tension force** in the rope. By implementing the constraint, we reduced the problem from two to only one degree of freedom. Furthermore, the tension in the rope does no work: it is always perpendicular to the motion of the bob, and hence the work contribution $\boldsymbol{T} \cdot d\boldsymbol{r} = 0$, where $\boldsymbol{T}$ is the tension force and $d\boldsymbol{r}$ is the displacement of the bob. Thus, our potential energy U — related to work done by a force — was simply the potential energy due to gravity alone, $U = mgh$. In general, whenever a contact force is always perpendicular to the displacement of the particle it is acting on, it can safely be ignored in constructing the Lagrangian.

EXAMPLE 4-2: A bead sliding on a vertical helix

A bead of mass m is slipped onto a frictionless wire wound in the shape of a helix of radius R, whose symmetry axis is oriented vertically in a uniform gravitational field, as shown in Figure 4.2. Using cylindrical coordinates r, θ, z, the base of the helix is located at $z = 0$, $\theta = 0$, and the angle θ is related to the height z at any point by $\theta = \alpha z$, where α is a constant with dimensions of inverse length. The gravitational potential energy of the bead is $U = mgz$, and its kinetic energy is $T = (1/2)mv^2 = (1/2)m[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2]$. However, the constraint that the bead slide along the helix tells us that the bead's radius is constant at $r = R$, and (choosing z as the single generalized coordinate), $\dot{\theta} = \alpha\dot{z}$. Therefore the kinetic energy of the bead is simply

$$\begin{aligned} T &= \frac{1}{2}m\,v^2 = \frac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\,(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2) \\ &= (1/2)m[0 + \alpha^2 R^2 + 1]\dot{z}^2\ , \end{aligned} \tag{4.19}$$

where we switched to cylindrical coordinates and implemented the *two* constraints $\theta = \alpha z$ and $r = R$. So the Lagrangian of the bead is

$$L = T - U = \frac{1}{2}m[1 + \alpha^2 R^2]\dot{z}^2 - mgz \tag{4.20}$$

in terms of the single generalized coordinate z and its generalized velocity $\dot{z}$. Two constraints reduced the dynamics from three to only one degree of freedom. Note that in Newtonian mechanics we often need to take into account the normal force of the wire on the bead as one of the forces in $\boldsymbol{F} = m\boldsymbol{a}$; however, the normal force appears nowhere in the Lagrangian, because it does no work on the bead in this case — it is always perpendicular to the displacement of the bead[4]. In general, whenever a normal force is perpendicular to the displacement of a particle it is acting on, we can safely ignore it in setting up the Lagrangian.

EXAMPLE 4-3: Block on an inclined plane

A block of mass m slides down a frictionless plane tilted at angle α to the horizontal, as shown in Figure 4.3. The gravitational potential energy is $mgh = mgX\sin\alpha$, where X is the distance of the block up along the plane from its lowest point. Using X as the generalized coordinate, the velocity is $\dot{X}$, and the Lagrangian of the block is

[4]In this simple case with a single normal force, the simplification is not very obvious. However, in more complicated scenarios we shall see later, the advantages of dropping the normal force from the Lagrangian — versus the Newtonian approach – will become more apparent.

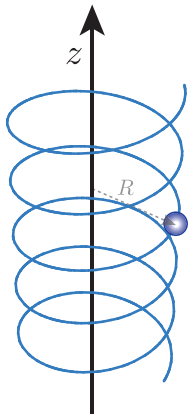

FIGURE 4.2 : A bead sliding on a vertically-oriented helical wire

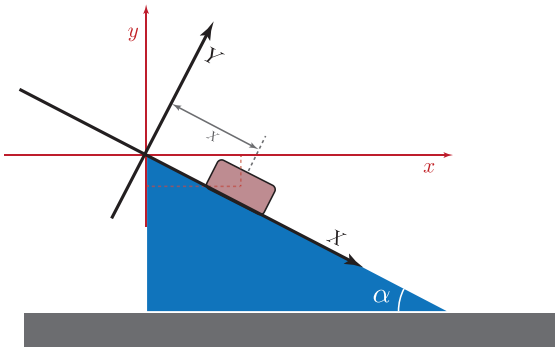

FIGURE 4.3 : Block sliding down an inclined plane

$$L = T - U = \frac{1}{2}mv^2 - U = \frac{1}{2}m\dot{X}^2 - mgX\sin\alpha, \tag{4.21}$$

which depends explicitly upon a single coordinate (X) and a single velocity ($\dot{X}$). Our single degree of freedom is then X. The Euler equation is

$$\frac{\partial L}{\partial X} - \frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = -mg\sin\alpha - \frac{d}{dt}m\dot{X} = 0, \quad \text{or} \quad -mg\sin\alpha = m\ddot{X}, \tag{4.22}$$

which is indeed the correct $\boldsymbol{F} = m\,\boldsymbol{a}$ equation for the block along the tilted plane.

In this problem, we judiciously chose our only degree of freedom as X, the distance along the inclined plane. We can think of this as a coordinate transformation from x,y as shown in the Figure, to X,Y. We then have a constraint $\dot{Y} = 0$, since the block cannot tunnel into the plane. We are then left with a single degree of freedom, X. This constraint is associated with the normal force. Furthermore, the normal force in this problem is always perpendicular to the displacement of the block and thus does no work. Once again, the Lagrangian formalism demonstrates its elegance and power by dropping from the computation a force – and a corresponding equation – and reducing the effective number of degrees of freedom. ———

4.3 Generalized momenta and cyclic coordinates

In Cartesian coordinates the kinetic energy of a particle is $T = (1/2)m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$, whose derivatives with respect to the velocity components are $\partial L/\partial\dot{x} = m\dot{x}$, etc., which are the components of momentum. So with generalized coordinates q_k, it is natural to define the **generalized momenta** p_k as

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}. \tag{4.23}$$

In terms of p_k, the Lagrange equations become simply

$$\frac{dp_k}{dt} = \frac{\partial L}{\partial q_k}. \tag{4.24}$$

Now sometimes a particular coordinate q_l is *absent* from the Lagrangian. Its generalized velocity $\dot{q}_l$ is present, but not q_l itself. A missing coordinate is said to be a **cyclic coordinate** or an **ignorable coordinate**.[5] For any

[5]Neither "cyclic" nor "ignorable" is a particularly appropriate or descriptive name for a coordinate absent from the Lagrangian, but they are nevertheless the conventional terms. In this book we will most often call any missing coordinate "cyclic".

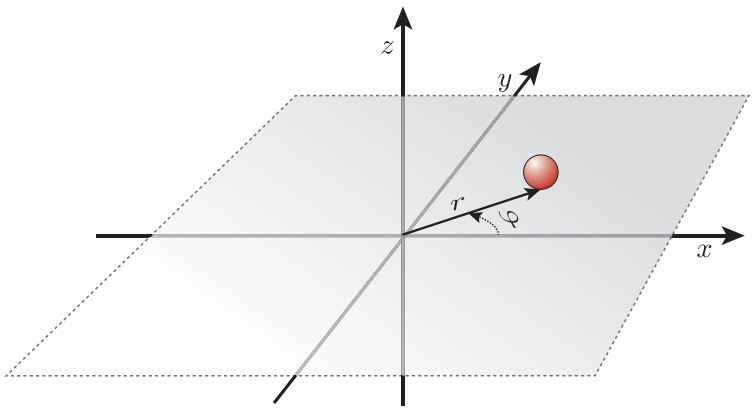

FIGURE 4.4 : Particle moving on a tabletop

such coordinate the Lagrange equation (4.24) tells us that the time derivative of the corresponding generalized momentum is zero, so that particular generalized momentum is *conserved*.

One of the first things to notice about a Lagrangian is whether there are any cyclic coordinates, because any such coordinate leads to a conservation law that is also a first integral of motion. This means that the equation of motion for that coordinate is already half solved, in that it is only a first-order differential equation rather than the second-order differential equation one typically gets for a noncyclic coordinate.

EXAMPLE 4-4: Particle on a tabletop, with a central force

For a particle moving in two dimensions, such as on a tabletop, it is often useful to use polar coordinates (r, φ) about some origin, as shown in Figure 4.4. The kinetic energy of the particle is

$$T = \frac{1}{2} m\, v^2 = \frac{1}{2} m\, (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m\, \left[\dot{r}^2 + (r\dot{\varphi}\right]^2). \tag{4.25}$$

Alternatively, we could have started in three dimensions in cylindrical coordinates with the addition of a $\dot{z}^2$, then use the constraint $\dot{z}^2 = 0$ that gets rid of the normal force being applied by the table onto the particle vertically. This force does no work and can be ignored using a constraint.

We will assume here that any force acting on the particle is a central force, depending upon r alone, so the potential energy U of the particle also depends upon r alone. The Lagrangian is therefore

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r). \tag{4.26}$$

Our two degrees of freedom are r and φ. We note right away that in this case the coordinate φ is cyclic, so there must be a conserved quantity

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}, \tag{4.27}$$

which we recognize as the **angular momentum** of the particle. In Lagrange's approach, p_φ is conserved because φ is a cyclic coordinate; in Newtonian mechanics, p_φ is conserved because there is no torque on the particle, since we assumed that any force is a central force. The various partial derivatives of L are

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \tag{4.28}$$

$$\frac{\partial L}{\partial r} = mr\dot{\varphi}^2 - \frac{\partial U(r,\varphi)}{\partial r} \qquad \frac{\partial L}{\partial \varphi} = 0, \tag{4.29}$$

so the Lagrange equations

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0 \qquad \text{and} \qquad \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0 \tag{4.30}$$

become

$$mr\dot{\varphi}^2 - \frac{\partial U(r)}{\partial r} - m\ddot{r} = 0 \qquad \text{and} \qquad -mr\ddot{\varphi} - 2m\dot{r}\dot{\varphi} = 0 \tag{4.31}$$

or (equivalently)

$$F_r = m(\ddot{r} - r\dot{\varphi}^2) \equiv ma_r \qquad \text{and} \qquad F_\varphi = m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) = 0, \tag{4.32}$$

where the radial force is $F_r = -\partial U/\partial r$ and the radial and tangential accelerations are

$$a_r = \ddot{r} - r\dot{\varphi}^2 \qquad \text{and} \qquad a_\varphi = r\ddot{\varphi} + 2\dot{r}\dot{\varphi}, \tag{4.33}$$

and where the tangential acceleration a_φ is zero in this case.[6]

In an example of Chapter 1 we found (using $\boldsymbol{F} = m\boldsymbol{a}$) the equations of motion of a particle of mass m on the end of a spring of zero natural length and force constant k, where one end

[6]Note how easy it is to get the expressions for radial and tangential accelerations in polar coordinates using this method. They are often found in classical mechanics by differentiating the position vector $\boldsymbol{r} = r\hat{\boldsymbol{r}}$ twice with respect to time, which involves rather tricky derivatives of the unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$.

of the spring was fixed and the particle was free to move in two dimensions, as on a tabletop. There we used Cartesian coordinates (x, y). Now we are equipped to write the equations of motion in polar coordinates instead. Equations (4.30) with $F_r = -kr$ and $F_\varphi = 0$ give

$$-kr = m(\ddot{r} - r\dot{\varphi}^2) \quad \text{and} \quad p_\varphi = mr^2\dot{\varphi} = \text{constant}. \tag{4.34}$$

That is, since the Lagrangian is independent of φ, we get the immediate first integral of motion $p_\varphi =$ constant. Eliminating $\dot{\varphi}$ between the two equations, we find the purely radial equation

$$\ddot{r} - \frac{(p_\varphi)^2}{m^2 r^3} + \omega_0^2 r = 0 \tag{4.35}$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency the spring-mass system would have if the mass were oscillating in one dimension (which in fact it would do if the angular momentum p_φ happened to be zero.) Note that even though the motion is generally two-dimensional, equation (4.35)) contains only $r(t)$; we can therefore find a first integral of this equation because it has the form of a one-dimensional $F = ma$ equation with $F = m\ddot{r} = (p_\varphi^2/mr^3 - m\omega_0^2 r$. An effective one-dimensional potential energy can be found by setting $F(r) = -dU_{\text{eff}}(r)/dr$; that is,

$$U_{\text{eff}}(r) = -\int^r F(r)\, dr = -\int^r \left(\frac{(p_\varphi)^2}{mr^3} - m\omega_0^2 r \right) dr = \frac{(p_\varphi)^2}{2mr^2} + \frac{1}{2}kr^2 \tag{4.36}$$

plus a constant of integration, which we might as well set to zero. Therefore, a first integral of motion is

$$\frac{1}{2}m\dot{r}^2 + \frac{(p_\varphi)^2}{2mr^2} + \frac{1}{2}kr^2 = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}} = \text{constant}. \tag{4.37}$$

A sketch of U_{eff} is shown in Figure 4.5. Note that U_{eff} has a minimum, which is the location of an equilibrium point (the value of r for which $dU_{\text{eff}}(r)/dr = 0$ is of course also the radius for which $\ddot{r} = 0$.) If r remains at the minimum of U_{eff}, the mass is actually circling the origin. The motion about this point is stable because the potential energy is a minimum there. For small displacements from equilibrium the particle oscillates back and forth about this equilibrium radius as it orbits the origin. In Section 4.7 we will calculate the frequency of these oscillations. This example demonstrates the use of coordinate transformations in the Lagrangian formalism. We already knew the formalism goes through under a coordinate change. In this case, we see how useful it can be to use this freedom at the Lagrangian level.

EXAMPLE 4-5: The spherical pendulum

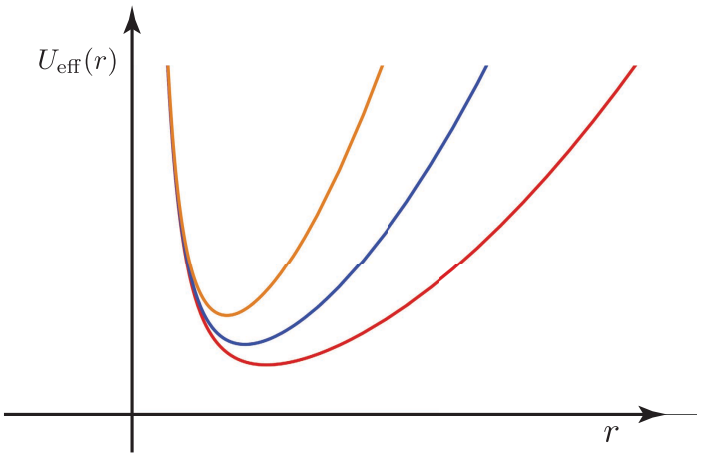

FIGURE 4.5 : The effective radial potential energy for a mass m moving with an effective potential energy $U_{\rm eff} = (p^\varphi)^2/2mr^2 + (1/2)kr^2$ for various values of p^φ, m, and k.

A ball of mass m swings on the end of an unstretchable string of length R in the presence of a uniform gravitational field g. This is often called the "spherical pendulum", because the ball moves as though it were sliding on the frictionless surface of a spherical bowl. We aim to find its equations of motion.

The ball has two degrees of freedom:

(i) It can move horizontally around a vertical axis passing through the point of support, corresponding to changes in its azimuthal angle φ. (On Earth's surface this would correspond to a change in *longitude*.)

(ii) It can also move in the polar direction, as described by the angle θ. (On Earth's surface this would correspond to a change in *latitude*.)

These angles are illustrated in Figure 4.6. In spherical coordinates, the velocity square appears as

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2 \Rightarrow R^2\dot{\theta}^2 + R^2\sin^2\theta\dot{\varphi}^2 \tag{4.38}$$

using the constraint $r = R$, which gets rid of the tension force and gets us to two degrees of freedom. We know once again that the tension force is always perpendicular to the displacement in this problem, and hence can be thrown away through the use of a constraint. The velocities in the θ and φ directions are $v_\theta = R\dot{\theta}$ and $v_\varphi = R\sin\theta\dot{\varphi}$, which are perpendicular to one another. Hence, the kinetic energy becomes

$$T = \frac{1}{2}m\,R^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) \tag{4.39}$$

The altitude h of the ball, measured from its lowest possible point, is $h = R(1 - \cos\theta)$, so the potential energy can be written

$$U = mgh = mgR(1 - \cos\theta). \tag{4.40}$$

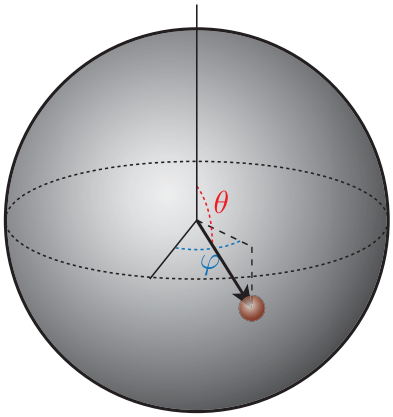

FIGURE 4.6 : Coordinates of a ball hanging on an unstretchable string

The Lagrangian is therefore

$$L = T - U = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) - mgR(1-\cos\theta)\ . \tag{4.41}$$

Our two degrees of freedom are then θ and φ. The derivatives are

$$\frac{\partial L}{\partial \theta} = mR^2\sin\theta\cos\theta\dot{\varphi}^2 - mgR\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta} \tag{4.42}$$

$$\frac{\partial L}{\partial \varphi} = 0 \qquad \frac{\partial L}{\partial \dot{\varphi}} = mR^2\sin^2\theta\ \dot{\varphi}. \tag{4.43}$$

The Lagrange equations become

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \qquad \text{and} \qquad \frac{\partial L}{\partial \varphi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 0, \tag{4.44}$$

but before writing them out, note that φ is cyclic, so the corresponding generalized momentum is

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mR^2\sin^2\theta\ \dot{\varphi} = \text{constant}, \tag{4.45}$$

which is an immediate first integral of motion. (We identify p_φ as the angular momentum about the vertical axis.)

Now since

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\ddot{\theta}, \tag{4.46}$$

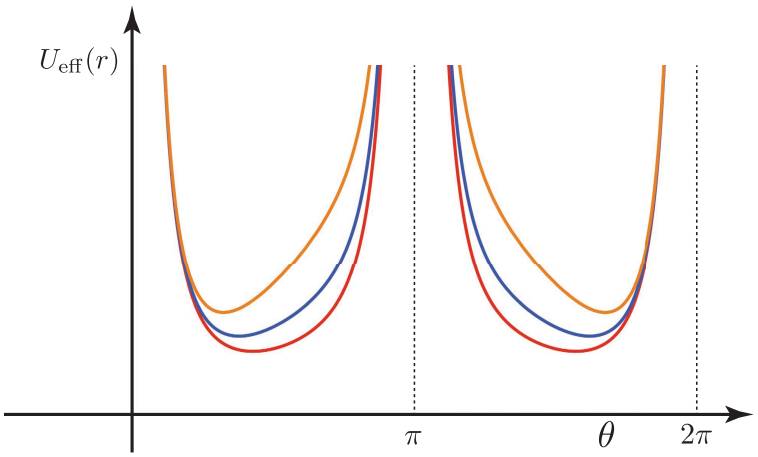

FIGURE 4.7 : A sketch of the effective potential energy $U_{\rm eff}$ for a spherical pendulum. A ball at the minimum of $U_{\rm eff}$ is circling the vertical axis passing through the point of suspension, at constant θ. The fact that there is a potential energy *minimum* at some angle θ_0 means that if disturbed from this value the ball will oscillate back and forth about θ_0 as it orbits the vertical axis.

the θ equation can be written

$$mR^2\ddot{\theta} = mR^2 \sin\theta\cos\theta\dot{\varphi}^2 - mgR\sin\theta, \tag{4.47}$$

so

$$\ddot{\theta} - \sin\theta\cos\theta\ \dot{\varphi}^2 + \left(\frac{g}{R}\right)\sin\theta = 0. \tag{4.48}$$

We can eliminate the $\dot{\varphi}^2$ term using $\dot{\varphi} = p_\varphi/(mR^2\sin^2\theta)$, to give

$$\ddot{\theta} - \left(\frac{p_\varphi}{mR^2}\right)^2 \frac{\cos\theta}{\sin^3\theta} + \left(\frac{g}{R}\right)\sin\theta = 0, \tag{4.49}$$

a second-order differential equation for the polar angle θ as a function of time.

To make further progress, do we have to tackle this differential equation head-on? Not if we can find a first integral instead! In fact, we have already identified one first integral, the conservation of angular momentum

$$mR^2\sin^2\theta\ \dot{\varphi} = p_\varphi = \text{constant} \tag{4.50}$$

about the vertical axis. Another first integral is energy conservation

$$E = T + U = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) + mgR(1-\cos\theta), \tag{4.51}$$

valid because no work is being done on the ball aside from the work done by gravity, which is already accounted for in the potential energy. By combining the two conservation laws we can eliminate $\dot{\varphi}$:

$$E = \frac{1}{2}mR^2\dot{\theta}^2 + \frac{(p_\varphi)^2}{2mR^2\sin^2\theta} + mgR(1-\cos\theta) \tag{4.52}$$

or

$$E = \frac{1}{2}mR^2\dot{\theta}^2 + U_{\text{eff}} \tag{4.53}$$

where the "effective potential energy" is

$$U_{\text{eff}} = \frac{(p_\varphi)^2}{2mR^2\sin^2\theta} + mgR(1-\cos\theta). \tag{4.54}$$

This effective potential energy U_{eff} is sketched in Figure 4.7; it includes the terms in E that depend only on position. The second term is the actual gravitational potential energy, while the first term is really a piece of the kinetic energy that has become a function of position only, thanks to angular momentum conservation.

Equations (4.50)) and (4.51 form a first-order differential equation that can be reduced to an integral by solving it for $\dot{\theta}$ and then separating the variables t and θ and integrating both sides. The result is, if we choose $t = 0$ when $\theta = \theta_0$,

$$t(\theta) = \sqrt{\frac{mR^2}{2}}\int_{\theta_0}^{\theta}\frac{d\theta}{\sqrt{(E-mgR) - (p_\varphi)^2/(2mR^2\sin^2\theta) + mgR\cos\theta}}. \tag{4.55}$$

Once again, the constraint reduce the number of degrees of freedom, from three to two in this case. And the associated tension constraint force does no work since it is perpendicular to the trajectory; and hence we need not include its contribution to the Lagragian. ———————

4.4 Systems of particles

So far we have been thinking about the motion of single particles only, described by at most three generalized coordinates and three generalized velocities. But often we want to find the motion of **systems of particles**, in which two or more particles may interact with one another, like two blocks on opposite ends of a spring, or several stars orbiting around one another. Can we still use the variational approach, by writing down a Lagrangian that contains the total kinetic energy and the total potential energy of the entire system?

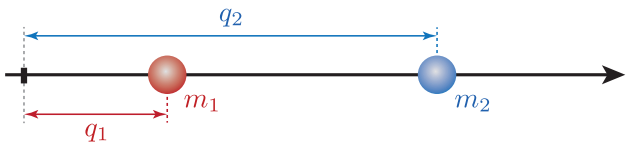

FIGURE 4.8 : Two interacting beads on a one-dimensional frictionless rail. The interaction between the particles depends only on the distance between them.

EXAMPLE 4-6: Two interacting particles

Consider a system of two particles, with masses m_1 and m_2, confined to move along a horizontal frictionless rail. Figure 4.8 shows a picture of the setup, where we label the coordinates of the particles x_1 and x_2. We can then write an action for the system

$$S = \int dt \left(\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - U(x_2 - x_1) \right) \tag{4.56}$$

where, in additional to the usual kinetic energy terms, there is some unknown interaction between the particles described by a potential $U(x_2 - x_1)$. Note that we use the *total* kinetic energy, and we assume that the potential – hence the associated force law – depends only upon the distance between the particles. We then have two equations of motion with two generalized coordinates, x_1 and x_2, so that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} = 0 \Rightarrow m_1 \ddot{x}_1 = -\frac{\partial U}{\partial x_1} \tag{4.57}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} = 0 \Rightarrow m_2 \ddot{x}_2 = -\frac{\partial U}{\partial x_2} = +\frac{\partial U}{\partial x_1} \tag{4.58}$$

where in the last step we used the fact that $U = U(x_2 - x_1)$. We see that we have a second law of Newton for each of the two particles: kinetic energy is additive and each of its terms will generally give the $m\boldsymbol{a}$ part of Newton's second law for the corresponding particle. Hence,

in multi-particle systems, we need to consider the *total* kinetic energy T minus the *total* potential energy. Terms that mix the variables of different particles, such as $U(x_2 - x_1)$, will give the correct forces on the particles as well. In this case, we see that the action-reaction pair, $\partial U/\partial x_1 = -\partial U/\partial x_2$, comes out for *free*, and arises from the fact that the force law depends only on the distance between the particles! That is, Newton's third law is naturally incorporated in the formalism and originates from the fact that forces between two particles depend only upon the distance between the interacting entities, and not (say) their absolute positions.

Suppose for example that the particles are connected by a Hooke's-law spring of force constant k. If we choose the coordinates x_1 and x_2 appropriately, the spring stretch will be $x_2 - x_1$, so the potential energy is $U = (1/2)k(x_2 - x_1)^2$. The Lagrange equations then give

$$m_1\ddot{x}_1 = +k(x_2 - x_1) \quad \text{and} \quad m_2\ddot{x}_2 = -k(x_2 - x_1). \tag{4.59}$$

The forces on the two particles are obviously equal but opposite: In such a case the total momentum of the systems must be conserved, which is easily verified simply by adding the two equations, to show that

$$\frac{d}{dt}(m_1\dot{x}_1 + m_2\dot{x}_2) = 0. \tag{4.60}$$

There are actually two conserved quantities in this problem, the momentum and the energy, each of which leads to a first integral of motion.

These results suggest that there must be a more transparent set of generalized coordinates to use here, in which one of the new coordinates is cyclic, so that its generalized momentum will be conserved automatically. These new coordinates are the **center of mass** and **relative** coordinates

$$X \equiv \frac{m_1x_1 + m_2x_2}{M} \qquad \text{and} \qquad x \equiv x_2 - x_1, \tag{4.61}$$

where $M = m_1 + m_2$ is the total mass of the system: Note that X and x are simply linear combinations of x_1 and x_2. Then in terms of X and x, it is straightforward to show that the Lagrangian of the system becomes

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}^2 - U(x) \tag{4.62}$$

where $\mu \equiv m_1m_2/M$ is called the **reduced mass** of the system (note that μ is in fact smaller than either m_1 or m_2.) Using this Lagrangian, it is obvious that the center of mass coordinate X is cyclic, so the corresponding momentum

$$P = \frac{\partial L}{\partial \dot{X}} = M\dot{X} \equiv m_1\dot{x}_1 + m_2\dot{x}_2 \tag{4.63}$$

is conserved (as we saw before).

This problem is an example of reducing a two-body problem to an equivalent one-body problem through a coordinate transformation. The motion of the center of mass of the system is trivial: the center of mass just drifts along at constant velocity. The interesting motion of

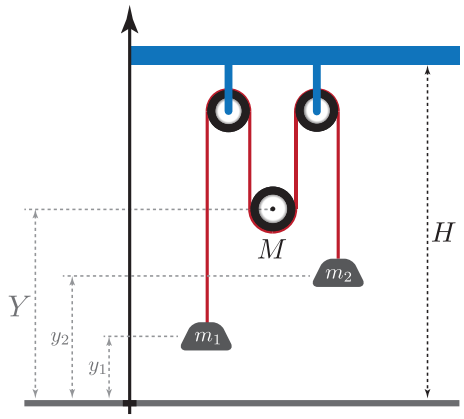

FIGURE 4.9 : A contraption of pulleys. We want to find the accelerations of all three weights. We assume that the pulleys have negligible mass so they have negligible kinetic and potential energies.

the particles is their *relative* motion x, which behaves as though it were a single particle of mass μ and position $x(t)$ subject to the potential energy $U(x)$ with Lagrangian

$$L = \frac{1}{2}\mu\dot{x}^2 - U(x) \ . \tag{4.64}$$

Hence, coordinate transforming at the Lagrangian level can be very powerful technique. ——

EXAMPLE 4-7: Pulleys everywhere

Another classic set of mechanics problems involves pulleys, lots of pulleys. Consider the setup shown in Figure 4.9. Two weights, with masses m_1 and m_2, hang on the outside of a three-pulley system, while a weight of mass M hangs on the middle pulley. We assume the pulleys and the connecting rope have negligible mass, so their kinetic and potential energies are also negligible. We will suppose for now that all three pulleys have the same radius R, but this will turn out to be of no importance. We want to find the accelerations of m_1, m_2, and M. We construct a Cartesian coordinate system as shown in the figure, which is at rest in an inertial frame of the ground. First of all, note that there are three massive objects moving in two dimensions, so we might think that we have six variables to track, x_1 and y_1 for weight m_1, x_2 and y_2 for weight m_2, and X and Y for weight M. We can then write the *total* kinetic energy

$$T = \frac{1}{2}m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2}m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right) + \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right). \tag{4.65}$$

But that is obviously overkill. We know the dynamics is entirely vertical, so we can focus on y_1, y_2, and Y *only* and set $\dot{x}_1 = \dot{x}_2 = \dot{X} = 0$. But that is still too much. There are only *two* degrees of freedom in this problem! Just pick any two of y_1, y_2, or Y, and we can draw the figure uniquely, as long as we know the length of the rope. Another way of saying this is to write

$$\text{Length of rope} = (H - y_1) + 2\,(H - Y) + (H - y_2) + 3\pi R, \tag{4.66}$$

where H is the height of the ceiling, as shown in the figure. We can therefore write in general

$$y_1 + 2\,Y + y_2 = \text{constant}\ , \tag{4.67}$$

which can be used to eliminate one of our three variables. We choose to get rid of Y, using

$$Y = -\frac{y_1 + y_2}{2} + \text{constant}, \tag{4.68}$$

which implies

$$\dot{Y} = -\frac{\dot{y}_1 + \dot{y}_2}{2}\ . \tag{4.69}$$

We can now write our kinetic energy in terms of two variables only, y_1 and y_2,

$$T = \frac{1}{2} m_1\, \dot{y}_1^2 + \frac{1}{2} m_2\, \dot{y}_2^2 + \frac{1}{2} M \left(\frac{\dot{y}_1 + \dot{y}_2}{2}\right)^2\ . \tag{4.70}$$

We next need the potential energy, which is entirely gravitational. We can write

$$\begin{aligned} U &= m_1 g\, y_1 + m_2 g\, y_2 + M\, g\, Y \\ &= m_1 g\, y_1 + m_2 g\, y_2 - M\, g \left(\frac{y_1 + y_2}{2}\right) + \text{constant} \end{aligned} \tag{4.71}$$

where the zero of the potential was chosen at the ground, and we can also drop the additive constant term since it does not affect the equations of motion. In summary we have a variational problem with a Lagrangian

$$\begin{aligned} L &= T - U = \frac{1}{2} m_1\, \dot{y}_1^2 + \frac{1}{2} m_2\, \dot{y}_2^2 + \frac{1}{2} M \left(\frac{\dot{y}_1 + \dot{y}_2}{2}\right)^2 \\ &- m_1 g\, y_1 - m_2 g\, y_2 + M g \left(\frac{y_1 + y_2}{2}\right). \end{aligned} \tag{4.72}$$

There are two dependent variables y_1 and y_2, so we have two equations of motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_1}\right) - \frac{\partial L}{\partial y_1} = 0 \Rightarrow m_1 \ddot{y}_1 + \frac{M}{4}\,(\ddot{y}_1 + \ddot{y}_2) = -m_1 g + \frac{M\,g}{2} \tag{4.73}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}_2}\right) - \frac{\partial L}{\partial y_2} = 0 \Rightarrow m_2 \ddot{y}_2 + \frac{M}{4}\,(\ddot{y}_1 + \ddot{y}_2) = -m_2 g + \frac{M\,g}{2}\ . \tag{4.74}$$

We can now solve for $\ddot{y}_1$ and $\ddot{y}_2$,

$$\begin{aligned} \ddot{y}_1 &= -g + \frac{4\,m_2 g}{m_1 + m_2 + 4\,m_1\,m_2/M} \\ \ddot{y}_2 &= -g + \frac{4\,m_1 g}{m_1 + m_2 + 4\,m_1\,m_2/M}\ . \end{aligned} \tag{4.75}$$

Note that these accelerations have magnitudes *less* than g, as we might expect intuitively. We can also find $\ddot{Y}$ from (4.69)

$$\ddot{Y} = -\frac{\ddot{y}_1 + \ddot{y}_2}{2} \Rightarrow \ddot{Y} = g - \frac{2\,(m_1 + m_2)g}{m_1 + m_2 + 4\,m_1\,m_2/M}\ . \tag{4.76}$$

The constraint given by (4.66), which eliminated one of our three original variables, implements the physical condition that the rope has constant length. This is related to the tension force in the rope. The astute reader may rightfully wonder whether we have overlooked something in this treatment: we never encountered the tension force of the rope on each of the masses! Consider the two tension forces T_1 and T_2 at the end of this (or any) massless rope. If we wanted to account for such forces in a Lagrangian, we would need the associated energy, or **work** they contribute to the system. Since the rope has zero mass, we know that $|T_1| = |T_2|$. The two tension forces however point in opposite directions. When one end of the rope moves by $\Delta x_1 > 0$ parallel to T_1, T_1 does work $W_1 = |T_1|\Delta x_1$. At the same time, the other end *must* move the same distance $\Delta x_2 = \Delta x_1$. However, at this other end, the tension force points opposite to the displacement, and the work is $W_2 = -|T_2|\Delta x_2 = -|T_1|\Delta x_1$. The *total* work is $W = W_1 + W_2 = 0$. Hence, the tension forces along a massless rope will always contribute zero work, and hence cannot be associated with energy in the Lagrangian. Similarly, this is the case for any force that appears in a problem in an action-reaction pair, as we shall see next.

EXAMPLE 4-8: A block on a movable inclined plane

Let us return to the classic problem of a block sliding down a frictionless inclined plane, as in Example 2, except we will make things a bit more interesting: now the inclined plane itself is allowed to move! Figure 4.10(a) shows the setup. A block of mass m rests on an inclined plane of mass M: Both the block and the inclined plane are free to move without friction. The plane's angle is denoted by α. The problem is to find the acceleration of the block.

The observation deck is the ground, which is taken as an inertial reference frame. We set up a convenient set of Cartesian coordinates, as shown in the figure. The origin is shifted to the top of the incline at zero time to make the geometry easier to analyze. We start by identifying the degrees of freedom. At first, we can think of the block and inclined plane as

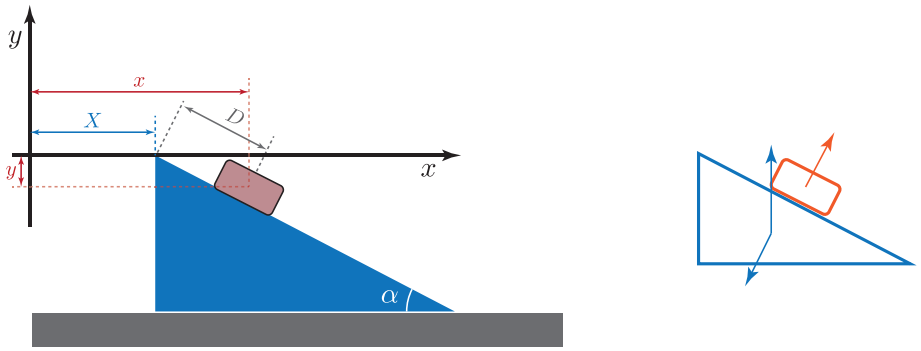

FIGURE 4.10 : A block slides along an inclined plane. Both block and inclined plane are free to move along frictionless surfaces.

moving in the two dimensions of the problem. The block's coordinates could be denoted by x and y, and the inclined plane's coordinates by X and Y. But we quickly realize that this would be overkill: if we specify X, and how far down the top of the incline the block is located (denoted by D in the figure), we can draw the figure uniquely. This is because the inclined plane cannot move vertically, either jumping off the ground or burrowing into it (that's one condition), so Y is unnecessary, and the horizontal position x of the block is determined by X and D (that's a second condition.) We then start with four coordinates, add two conditions or restrictions, and we are left with *two* degrees of freedom. The choice of the two remaining degrees of freedom is arbitrary, as long as the choice uniquely fixes the geometry. We will pick X and D; another choice might be X and x, for example.

Next, we need to write the Lagrangian. The starting point for this is the total kinetic energy of the system,

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}M\left(\dot{X}^2 + \dot{Y}^2\right) \tag{4.77}$$

in the inertial frame of the ground. Note the importance of writing the kinetic energy in an inertial frame, even if it means using more coordinates than the generalized coordinates that will be used in the Lagrangian.

Now we need to rewrite the kinetic energy in terms of the two degrees of freedom X and D alone. This requires a little bit of geometry. Looking back at the figure, we can write

$$Y = 0\ , \qquad x = X + D\cos\alpha\ , \qquad y = -D\sin\alpha\ . \tag{4.78}$$

This implies

$$\dot{Y} = 0 \quad , \quad \dot{x} = \dot{X} + \dot{D}\cos\alpha \quad , \quad \dot{y} = -\dot{D}\sin\alpha\ . \tag{4.79}$$

We can now substitute these into (4.77) and get

$$T = \frac{1}{2}M\,\dot{X}^2 + \frac{1}{2}m\,\dot{X}^2 + \frac{1}{2}m\,\dot{D}^2 + m\,\dot{X}\,\dot{D}\,\cos\alpha\ . \tag{4.80}$$

Note that this result, in terms of the generalized coordinates and velocities, would have been very difficult to guess, especially the $\dot{X}\dot{D}$ term. Again, it is very important to start by writing the kinetic energy first in an inertial frame, and often important as well to use Cartesian coordinates in this initial expression, to be confident that it has been done correctly. We now need the potential energy of the system, which is entirely gravitational. The inclined plane's potential energy does not change. Since it is a constant, we need not add it to the Lagrangian: the Lagrange equations of motion involve partial derivatives of L and, hence, a constant term in L is irrelevant to the dynamics. The block's potential energy on the other hand *does* change. We can choose the zero of the potential at the origin of our coordinate system and write

$$U = m\,g\,y = -m\,g\,D\,\sin\alpha\ . \tag{4.81}$$

The Lagrangian is now

$$L = T - U = \frac{1}{2}M\,\dot{X}^2 + \frac{1}{2}m\,\dot{X}^2 + \frac{1}{2}m\,\dot{D}^2 + m\,\dot{X}\,\dot{D}\,\cos\alpha + m\,g\,D\,\sin\alpha \tag{4.82}$$

We observe immediately that X is cyclic, so its corresponding momentum is conserved; also the total energy is conserved. Therefore we can obtain the complete set of two first integrals of motion.

Nevertheless, to illustrate a different approach, we will tackle the full second-order differential equations of motion obtained directly from the Lagrange equations. Since we have two degrees of freedom X and D, we'll have two second-order equations. The equation for X is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{X}}\right) - \frac{\partial L}{\partial X} = 0 \Rightarrow (m+M)\ddot{X} + m\,\ddot{D}\cos\alpha = 0, \tag{4.83}$$

and the equation for D is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{D}}\right) - \frac{\partial L}{\partial D} = 0 \Rightarrow m\,\ddot{D} + m\,\ddot{X}\cos\alpha = m\,g\,\sin\alpha. \tag{4.84}$$

This is a system of two linear equations in two unknowns $\ddot{X}$ and $\ddot{D}$. The solution is

$$\ddot{X} = \frac{g\,\cos^2\alpha\,\sin\alpha}{(1+M/m)\left(\sin^2\alpha + M/m\right)} \qquad \ddot{D} = -\frac{g\,\sin\alpha\cos\alpha}{\sin^2\alpha + M/m}. \tag{4.85}$$

Since we want the acceleration of the block in our inertial reference frame, we need to find $\ddot{x} \equiv a_x$ and $\ddot{y} \equiv a_y$. Differentiating (4.79) with respect to time, we get

$$a_x = \ddot{x} = \ddot{X} + \ddot{D}\,\cos\alpha \quad , \quad a_y = \ddot{y} = -\ddot{D}\,\sin\alpha. \tag{4.86}$$

Substituting our solution from (4.85) into these, we have

$$a_x = -\frac{(M/m)g\,\cos^2\alpha\,\sin\alpha}{(1+M/m)\left(\sin^2\alpha + M/m\right)} \qquad a_y = \frac{g\,\sin^2\alpha\cos\alpha}{\sin^2\alpha + M/m}. \tag{4.87}$$

It is always useful to look at various limiting cases, to see if a result makes sense. For example, what if $\alpha = 0$, *i.e.*, what if the block moves on a horizontal plane? Both accelerations then vanish, as expected: if started at rest, both block and incline just stay put. Now what if the inclined plane is much heavier than the block, with $M \gg m$? We then have

$$a_x \simeq -\frac{m}{M}\, g\, \cos^2\alpha\, \sin\alpha \qquad a_y \simeq \frac{m}{M} g\, \sin^2\alpha \cos\alpha, \tag{4.88}$$

so that $a_y/a_x \simeq -\tan\alpha$, which is what we would expect if the inclined plane were not moving appreciably.

The computational step at the beginning where we zeroed onto the degrees of freedom of the problem – going from X, Y, x, and y to X and D – is associated with the normal forces. The most impressive aspect of this example is the absence of any normal forces from our computations! With the traditional approach of problem solving, we would need to include several normal forces in the computation, shown in Figure 4.10(b): the normal force exerted by the inclined plane on the block, the normal force exerted by the ground on the inclined plane, *and* the normal force exerted by the block on the inclined plane as a reaction force. The role of these normal forces is to hold the inclined plane on the ground and to hold the block on the inclined plane. If we think of the contributions of the normal forces to the Lagrangian, we would want to include some potential energy terms for them. But potential energy is related to work done by forces. The normal force is often perpendicular to the direction of motion, and hence does no work, $\boldsymbol{N} \cdot \Delta\boldsymbol{r} = 0$. Hence, there is no potential energy term to include in the Lagrangian to account for such normal forces. In our example, this is not entirely correct... While it is true for the normal force exerted by the ground onto the inclined plane, it is *not* true for the normal forces acting between the block and inclined plane. This is because the inclined plane is moving as well and the trajectory of the block is not parallel to the incline! However, there is another reason why this normal force is safely left out of the Lagrangian method. These normal forces occur as an action-reaction pair. And the displacement of the interface between the block and incline is the same for both forces, and hence the contributions to the total work or energy of the system from these two normal forces cancel. As we saw from the previous example, such forces do not appear in the Lagrangian.

In a later chapter we will also learn of a way to impose the inclusion of normal and tension forces in a Lagrangian even when we need not do so — for the purpose of finding the magnitude of a normal force if it is desired. For now, we are very happy to drop normal and tension forces from consideration. This can be a big simplification for problem solving: fewer variables, fewer forces to consider, less work to do (no pun intended). ———————

4.5 The Hamiltonian

We will now prove an enormously useful mathematical consequence of the Lagrange equations, providing one more potential way to achieve a first integral of motion, which we can add to the arsenal of approaches already

summarized in the preceding section. First, take the total derivative of the Lagrangian $L(t, q_k, \dot{q}_k)$ with respect to time t. There are many ways in which L can change: it can change because of *explicit* changes in t, and also because of *implicit* changes in t due to the time dependence of one or more of the coordinates $q_k(t)$ or velocities $\dot{q}_k(t)$. Therefore, from multivariable calculus,

$$\frac{dL(q_k, \dot{q}_k, t)}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_k}\dot{q}_k + \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k, \tag{4.89}$$

using the Einstein summation convention from Chapter 2. That is, since the index k is repeated in each of the last two terms, a sum over k is implied in each term; we have also used the fact that $d\dot{q}_k/dt \equiv \ddot{q}_k$. Now, take the time derivative of the quantity $\dot{q}_k(\partial L/\partial \dot{q}_k)$; again, sums over k are implied:

$$\frac{d}{dt}\left(\dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) = \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \dot{q}_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \dot{q}_k \frac{\partial L}{\partial q_k} \tag{4.90}$$

using the product rule. We have also used the Lagrange equations to simplify the second term on the right. Note that this expression contains the same two summed terms that we found in equation (4.89)). Therefore, subtracting equation (4.90) from equation (4.89 gives

$$\frac{\partial L}{\partial t} - \frac{d}{dt}\left(L - \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) = 0. \tag{4.91}$$

This result is particularly interesting if L is not an *explicit* function of time, *i.e.*, , if $\partial L/\partial t = 0$. In fact, define the **Hamiltonian** H of a particle to be

$$H \equiv \dot{q}_k \; p_k - L \tag{4.92}$$

where we have already defined the generalized momenta to be $p_i = \partial L/\partial \dot{q}_k$, and again a sum over k is implied. Then equation (4.91) can be written

$$\frac{\partial L}{\partial t} = -\frac{dH}{dt}, \tag{4.93}$$

an extremely useful result! It shows that *if a Lagrangian L is not an explicit function of time, then the Hamiltonian H is conserved.*

What is the *meaning* of H? Suppose that our particle is free to move in three dimensions in a potential $U(x, y, z)$ without constraints, and that

we are using Cartesian coordinates. Then $p_x = m\dot{x}$, etc., so $\sum_i \dot{q}_k p_k = m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$. Therefore,

$$\begin{aligned} H &= m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z) \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z) = T + U = E, \end{aligned} \tag{4.94}$$

which is simply the energy of the particle!

Is H *always* equal to $E = T + U$? The answer is *no*, although very often it is. The precise conditions for which $H \neq E$ are derived in Appendix A.

EXAMPLE 4-9: Bead on a rotating parabolic wire

Suppose we bend a wire into the shape of a vertically oriented parabola defined by $z = \alpha r^2$, as illustrated in Figure 4.11, where z is the vertical coordinate and r is the distance of a point on the wire from the vertical axis of symmetry. Using a synchronous motor, we can force the wire to spin at constant angular velocity ω about its symmetry axis. Then we slip a bead of mass m onto the wire and find its equation of motion, assuming that it slides without friction along the wire.

We first have to choose generalized coordinate(s) for the bead. The bead moves in three dimensions, but because of the constraint we need only a single generalized coordinate to specify the bead's position. For example, if we know the distance r of the bead from the vertical axis of symmetry, we also know its altitude z, because it is constrained to move along the parabolic wire. And the bead also has no freedom to choose its azimuthal angle, because the synchronous motor turns the wire around at a constant rate, so given its angle φ_0 at time $t = 0$, its angle at other times is constrained to be $\varphi = \varphi_0 + \omega t$. So it is convenient to choose the cylindrical coordinate r as the generalized coordinate, although we could equally well choose the vertical coordinate z.

In cylindrical coordinates the square of the bead's velocity is the sum of squares of the velocities in the r, φ, and z directions,

$$v^2 = \dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2 = \dot{r}^2 + r^2\omega^2 + (2\alpha r\dot{r})^2 \tag{4.95}$$

since $\dot{\varphi} = \omega =$ constant, and $z = \alpha r^2$ for the parabolic wire. The gravitational potential energy is $U = mgz = mg\alpha r^2$, so the Lagrangian is

$$L = T - U = \frac{1}{2}m[(1 + 4\alpha^2 r^2)\dot{r}^2 + r^2\omega^2] - mg\alpha r^2. \tag{4.96}$$

We implemented two constraints, $\dot{\varphi} = \omega$ and $z = \alpha r^2$, and thus reduced the problem from three variables to one degree of freedom r. The contact force that keeps the bead on the wire

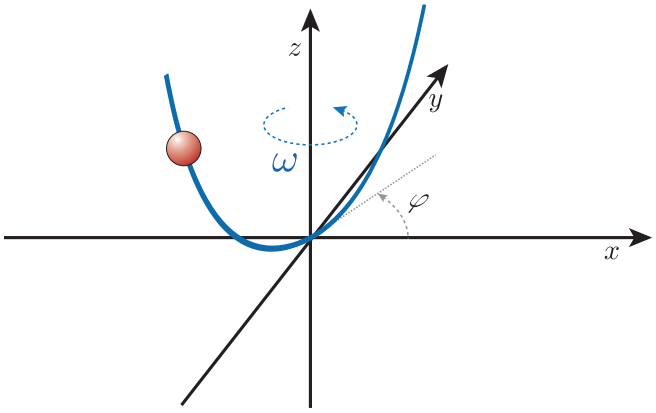

FIGURE 4.11 : A bead slides without friction on a vertically-oriented parabolic wire that is forced to spin about its axis of symmetry.

is the normal force associated with these two constraints. And we thus have chosen *not* to include its contribution to the Lagrangian assuming that this normal force has no contribution to the energy of the system and can be packaged into the two constraints safely... That this is correct to do is far from obvious in this setup since this normal force *does* work! This work is associated with the energy input by the motor to keep the wire turning at constant rate ω. Let us proceed anyways with this arrangement, and revisit the issue at the end of the section — hopefully justifying the absence of contributions from the normal force to our Lagrangian. The partial derivatives $\partial L/\partial \dot{r}$ and $\partial L/\partial r$ are easy to find, leading to the Lagrange equation

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial r} = m[4\alpha^2 r\dot{r}^2 + r\omega^2 - 2g\alpha r] - m\frac{d}{dt}(1+4\alpha^2 r^2)\dot{r} = 0 \tag{4.97}$$

which results finally in a second-order differential equation of motion

$$(1+4\alpha^2 r^2)\ddot{r} + 4\alpha^2 r\dot{r}^2 + (2g\alpha - \omega^2)r = 0. \tag{4.98}$$

Are we stuck with having to solve this second-order differential equation? Are there no first integrals of motion? The coordinate r is not cyclic, so p_r is not conserved. However, note that L is *not an explicit function of time*, so the Hamiltonian H *is* conserved! Conservation of H — which we will refer to as energy by definition — will give us a first integral of motion, so we are rescued; we do not have to solve the second-order differential equation (4.98) after all.

The generalized momentum is $p_r = \partial L/\partial \dot{r} = m(1+r\alpha^2 r^2)\dot{r}$, so the Hamiltonian is

$$\begin{aligned} H &= \dot{r}p_r - L = m(1+4\alpha^2 r^2)\dot{r}^2 - \frac{1}{2}m[(1+4\alpha^2 r^2)\dot{r}^2 + r^2\omega^2] + mg\alpha r^2 \\ &= \frac{1}{2}m[(1+4\alpha^2 r^2)\dot{r}^2 - r^2\omega^2] + mg\alpha r^2 = \text{constant}, \end{aligned} \tag{4.99}$$

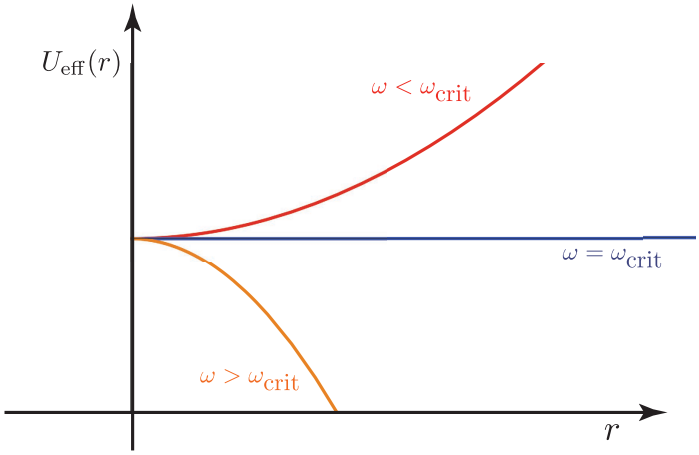

FIGURE 4.12 : The effective potential U_{eff} for the Hamiltonian of a bead on a rotating parabolic wire with $z = \alpha r^2$, depending upon whether the angular velocity ω is less than, greater than, or equal to $\omega_{\text{crit}} = \sqrt{2g}\ \alpha$.

which differs from the energy

$$E = T + U = \frac{1}{2}m[(1 + 4\alpha^2 r^2)\dot{r}^2 + r^2\omega^2] + mg\alpha r^2 \tag{4.100}$$

by $H - E = -mr^2\omega^2$.

Equation (4.99) is a first-order differential equation, which can be reduced to quadrature. Without going that far, we can understand a good deal about the motion just by using equation (4.99), and noting that it has a similar mathematical *form* to that of energy-conservation equations. That is, rewrite the equation as

$$H = \frac{1}{2}m(1 + 4\alpha^2 r^2)\dot{r}^2 + U_{\text{eff}}, \tag{4.101}$$

where

$$U_{\text{eff}} = \frac{1}{2}m(-r^2\omega^2) + mg\alpha r^2 = \frac{1}{2}mr^2(2g\alpha^2 - \omega^2). \tag{4.102}$$

This effective potential is quadratic in r with the interesting feature that everything depends upon how the angular velocity ω of the wire is related to a critical angular velocity $\omega_{\text{crit}} \equiv (2g)^{1/2}\ \alpha$, as illustrated in Figure 4.12. If $\omega < \omega_{\text{crit}}$, then U_{eff} rises with r, so the bead is stable at $r = 0$, the potential minimum. But if $\omega > \omega_{\text{crit}}$, then U_{eff} falls off with increasing r, so $r = 0$ is an unstable equilibrium point in that case; if the bead wanders even slightly from $r = 0$ at the bottom of the parabola it will be thrown out indefinitely far. The stability is neutral if $\omega = \omega_{\text{crit}}$, meaning that if placed at rest at any point along the wire the bead will stay at that point indefinitely, but if placed at any point and pushed outward it will keep moving outward, or if pushed inward it will keep moving inward.

This example shows that although the Hamiltonian function H is often equal to $E \equiv T+U$, this is not always so. Appendix A at the end of the chapter explains when and why they can differ. In any case, the Hamiltonian can be very useful, because it provides a first integral of motion if L is not explicitly time-dependent. It is also the starting point for an alternative approach to classical dynamics, as we will see in a later chapter, and it turns out to be an important bridge between classical and quantum mechanics[7].

Let us come back to the issue of dropping the normal force's contribution to our Lagrangian. Since the bead is sliding along the wire while rotating with it, and since the normal force is some vector perpendicular to the *wire* at any instant in time, we can see that this normal force is not necessarily perpendicular to the bead's displacement. Hence, it can have non-zero contribution to the energy of the system. This is why E is not conserved in the problem; the conserved quantity is H which is not equal to E in this case. Nevertheless, how can we justify dropping the normal force that we know can do work and hence may perhaps have a piece of U in $L = T - U$?

The answer is a delicate one: in constructing a Lagrangian, we need to include *all* objects that are moving around. The full system is bead *plus* rotating parabolic wire. The parabolic wire is not a free dynamical object since its motion is prescribed externally through the action of the motor. You can think of such a non-dynamical object as one with zero mass; *i.e.*, , it has no contribution to the kinetic energy of the system. In reality, it has a constant non-zero contribution, but a constant term in a Lagrangian does not effect the equations of motion. Hence, for simplicity, we can drop such a term altogether. However, forces can still act on it and do work. In this case, there is a normal force — equal but opposite in direction to the one acting on the bead — acting on the wire. The point of action of this normal on the wire is displaced by the same amount as the bead. Hence, the contribution to the work of the system of this normal force is equal in magnitude but opposite in sign to the contribution coming from the normal force acting on the bead. This is simply a reflection of Newton's third law: for every action, there is an equal but opposite reaction. The net contribution to work from these two normal forces adds up to zero. Hence, when we write U for the system, we only need to consider the contribution from gravity! We have $L = T - U$ as before, without a trace of the normal forces.

Once again, the Lagrangian formalism avoids dealing with contact forces and accounts for them through constraints – simplifying the problem significantly. We leave it as an exercise for the reader to solve this same problem using traditional force body diagram methods so as to appreciate the power of the Lagrangian formalism. ———————————

[7] In a different approach we will see in Chapter 6, conserved quantities are *defined* through associated symmetries in Nature. Within such a convention, the Hamiltonian would be defined as energy, and the combination $T + U$ loses its privileged name and place.

4.6 The moral of constraints

Let us summarize the steps we have used so far in setting up and preparing to solve Lagrange's equations.

1. We identify the degrees of freedom of each particle or object consistent with any constraints, and choose an appropriate set of generalized coordinates q_k for each.

2. Write the square of the velocity for each particle in terms of any convenient coordinates, usually Cartesian coordinates, in *some inertial reference frame*. Then reexpress the kinetic energy in terms of the generalized coordinates q_k, the generalized velocities $\dot{q}_k$, and the time if needed; *i.e.*, , $v^2 = v^2(q_k, \dot{q}_k, t)$. Then write the *total* kinetic energy T in terms of these v's.

3. Write the *total* potential energy in the form $U = U(q_k, t)$. Do *not* include any contributions from constraint forces.

4. Write the Lagrangian $L(t, q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...) = T - U$

5. Identify any cyclic coordinates in L; that is, identify any coordinate q_l missing in the Lagrangian, even though its corresponding generalized velocity $\dot{q}_l$ is present. In this case the corresponding generalized momentum $p_l \equiv \partial L/\partial \dot{q}_l$ is *conserved*. This gives a highly-valued first integral of motion, *i.e.*, , a differential equation that is first order rather than second order.

6. If there are more generalized coordinates in the problem than first integrals identified in the preceding steps, then one or more of the Lagrange equations of motion

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0. \tag{4.103}$$

   must be used as well, to obtain a complete set of differential equations. That is, if there are N generalized coordinates, we will generally need N mutually independent differential equations whose solutions will give the coordinates as functions of time. Some of these may be first-order equations, each corresponding to a conserved quantity, while others may be second-order equations.

All these steps but one were rigorously justified: that the Lagrangian $L = T - U$ accounts for all conservative forces, that coordinate transformations at the Lagrangian level are justified and in fact very useful, and that constraints assure that the generalized coordinates are independent and a reduced set of degrees of freedom describes the dynamics fully. But what about dropping constraint forces from the Lagrangian? We saw example after example that this works out fine. But now we see the emerging pattern in general. Constraint forces implement restrictions on the dynamics between *two objects in contact*. If both objects in question are part of the dynamical system (*i.e.*, , they both have kinetic energy contributions to the Lagrangian), we know that these constraint forces must come in equal and opposite pairs. Since the contact point is the same, this always implies that such forces will not do work and hence need not appear in the Lagrangian. On the other hand, if only one of the two objects is part of the dynamical system, the other one must then have *prescribed* time evolution by definition: *i.e.*, , the ground just sits there as a function of time, the pivot of the pendulum is fixed in position, a parabolic wire – on which a bead is sliding – is rotating at a given constant angular speed driven by a motor. In some of these cases, the constraint forces do no work because they are perpendicular to the displacement. But it is easy to see this statement in more generality: Extend the Lagrangian to include the non-dynamical system – the ground, the parabolic wire connected to a motor – by adding their *constant* kinetic energies to the Lagrangian. Then the constraint force becomes part of an internal action-reaction pair, which we know does not contribute to the Lagrangian! And the cost of adding the kinetic energy of the external agent to the Lagrangian is irrelevant: it is a constant shift to the Lagrangian since the relevant dynamics is, by definition, prescribed. This, we now see one of the central advantages of the Lagrangian formalism: drop all constraint forces from the outset!

As one gets used to the steps outlined above, many stages of this algorithmic process become second nature and can be done mentally. With practice, you may be able to stare at a complex mechanical system, write down the Lagrangian immediately on a single line, and in a few more lines, write the equations of motion! To get there however, one needs to first practice the steps summarized here ad nauseum, problem after problem.

4.7 Small oscillations about equilibrium

When we look around at many common mechanical systems, we find that energy is often approximately conserved and the system is in a more or less stable equilibrium state. For example, a nearby chair is resting on the floor, happily doing nothing as expected from a chair. It is in its minimal energy configuration. If we bump it, it wobbles a bit for some time, and then quickly finds itself again at rest in some new equilibrium state. When we bumped the chair, we added energy to the system, and the chair eventually dissipated this energy through friction (sound, heat, etc...) and found another minimal-energy motionless state.

In general, most mechanical systems can be accorded an energy of the form

$$\text{Constant} \times \dot{q}_k^2 + U_{\text{eff}}(q) = E \tag{4.104}$$

where the q_k's are the generalized coordinates and U_{eff} is an effective potential. We saw this in example after example in this chapter. For simplicity, imagine we have only one such coordinate we'll call q_k. If the effective potential energy U_{eff} has an extremum at some particular point $(q_k)_0$, then that point is an equilibrium point of the motion, so if placed at rest at $(q_k)_0$ the particle will stay there. If $(q_k)_0$ happens to be a *minimum* of U_{eff}, as illustrated in Figure 4.13, $(q_k)_0$ is a *stable* equilbrium point, so that if the particle is displaced slightly from $(q_k)_0$ it will oscillate back and forth, never wandering far from that point.

It is sometimes interesting to find the frequency of small oscillations about an equilibrium point. We can do this by fitting the bottom of the effective energy curve to a parabolic bowl, because that is the shape of the potential energy of a simple harmonic oscillator. That is, by the Taylor expansion

$$U(x) = U(x_0) + \frac{dU}{dx}|_{\text{x}_0}(x - x_0) + \frac{1}{2!}\frac{d^2U}{dx^2}|_{\text{x}_0}(x - x_0)^2 + ... \tag{4.105}$$

So if x_0 is the equilibrium point, by definition the second term vanishes, and the third term has the form $(1/2)k_{\text{eff}}(x - x_0)^2$, like that for a simple harmonic oscillator with center at x_0, where the effective force constant is given by $k_{\text{eff}} = U''(x_0)$. The frequency of small oscillations is therefore

$$\omega = \sqrt{k_{\text{eff}}/m} = \sqrt{U''(x_0)/m}. \tag{4.106}$$

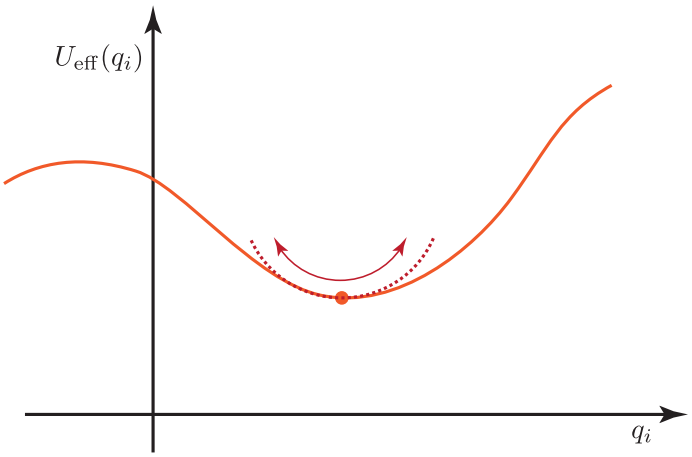

FIGURE 4.13 : An effective potential energy U_{eff} with a focus near a minimum. Such a point is a *stable* equilibrium point. The dotted parabola shows the leading approximation to the potential near its minimum. As the energy drains out, the system settles into its minimum with the final moments being well approximated with harmonic oscillatory dynamics.

Note that this explains the pervasiveness of the harmonic oscillator in Nature: since system will try to find their lowest energy configurations by dissipating energy, they will often find themselves near the minima of their effective potential. As we just argued, in the vicinity of such minima, systems will generically oscillate harmonically. An example will demonstrate how this works.

EXAMPLE 4-10: Particle on a tabletop with a central spring force

In Example 4 we considered a particle moving on a frictionless tabletop, subject to a central Hooke's-law spring force. There is an equilibrium radius for given energy and angular momentum for which the particle orbits in a circle of some radius r_0. Now we can find the oscillation frequency ω for the mass about the equilibrium radius if it is perturbed slightly from this circular orbit.

The effective potential in Example 4 was $U_{\text{eff}} = (p_\varphi)^2/2mr^2 + (1/2)kr^2$; the first derivative of this potential is $U'_{\text{eff}}(r) = -(p_\varphi)^2/mr^3 + kr$. The equilibrium value of r is located where $U'(r) = 0$; namely, where $r = r_0 = ((p_\varphi)^2/mk)^{1/4}$. The second derivative of $U_{\text{eff}}(r)$ is

$$U''(r) = 3(p_\varphi)^2/mr^4 + k, \tag{4.107}$$

so

$$U''(r_0) = \frac{3(p_\varphi)^2}{m((p_\varphi)^2/mk)} + k = 3k + k = 4k. \tag{4.108}$$

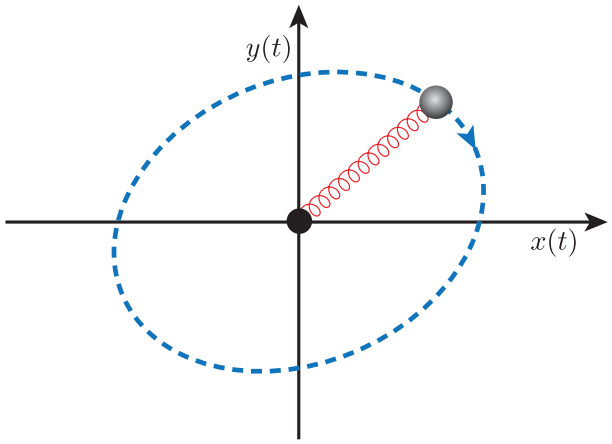

FIGURE 4.14 : The shape of the two-dimensional orbit of a particle subject to a central spring force, for small oscillations about the equilibrium radius.

The frequency of small oscillations about the equilibrium radius r_0 is therefore

$$\omega = \sqrt{U''(r_0)/m} = \sqrt{4k/m} = 2\omega_0. \tag{4.109}$$

That is, for the mass orbiting the origin and subject to a central Hooke's law spring force, the frequency of small oscillations about a circular orbit is just twice what it would be for the mass if it were oscillating back and forth in one dimension.

We can also find the *shape* of the orbits if the radial oscillations are small. The angular frequency of rotation is $\omega_{\rm rot} = v_\phi/r_o = (p_\varphi/mr_0)/r_0 = p_\varphi/(mr_0^2)$, where v_ϕ is the tangential component of velocity. But the equilibrium radius is $r_0 = ((p_\varphi)^2/mk)^{1/4}$, so the angular frequency of rotation is $\omega_{\rm rot} = p_\varphi/(mr_0^2) = p_\varphi/[m\sqrt{(p_\varphi)^2/k}] = \sqrt{k/m} = \omega_0$, which is the same as the frequency of oscillation of the system as if it were moving in one dimension! Therefore the frequency of radial oscillations (4.109) is just twice the rotational frequency, so the orbits for small oscillations are *closed*: that is, the path retraces itself in every orbit, as shown in Figure 4.14. The small-oscillation path appears to be elliptical, and in fact it is *exactly* elliptical, even for large displacements from equilibrium, as we already saw in Chapter 1 using Cartesian coordinates.

4.8 Relativistic generalization

By now, we have established the power of the Lagrangian formalism in dealing with mechanics problems involving one or more particles, interacting with

conservative forces, with or without a large class of constraints. But all this was within Newtonian mechanics — valid at speeds much less than the speed of light. Can we use the Lagrangian formalism for situations requiring relativistic treatment? The answer is rather simple to find. Our original setup for deriving the Lagrangian formalism started by integrating proper time to construct the action functional (see equation (3.67))). We then took the limit of low speeds in (3.73 to identify a piece of the future Lagrangian — the kinetic energy. Through the example of a particle in a uniform gravitational field, we identified the second piece in $L = T - U$ — the potential energy. We then proceeded to show that this works for any conservative potential, with one or more particles, with or without constraints. Going back to the beginning, we then must have

$$L = T - U = m\,c^2\,\sqrt{1 - \frac{v^2}{c^2}} - U \tag{4.110}$$

to tackle a fully relativistic problem. We can even still write $L = T - U$ using relativistic kinetic energy

$$T = m\,c^2\,\sqrt{1 - \frac{v^2}{c^2}} - m\,c^2 \tag{4.111}$$

since the additional piece $m\,c^2$ is a constant and hence does not effect the equations of motion. For a single particle in Cartesian coordinates, using our results from equations (3.72)), the equations of motion from (4.110 now look like

$$\frac{d}{dt}\,(\gamma\dot{x}) = -\frac{\partial U}{\partial x} \quad , \quad \frac{d}{dt}\,(\gamma\dot{y}) = -\frac{\partial U}{\partial y} \quad , \quad \frac{d}{dt}\,(\gamma\dot{z}) = -\frac{\partial U}{\partial z}\ . \tag{4.112}$$

This is the expected relativistic form of Newton's second law from (2.93)) or (2.100) if we identify $\boldsymbol{F} = -\boldsymbol{\nabla}U$. Hence, our entire Lagrangian formalism extends through the relativistic regime as long as we replace kinetic energy in $L = T - U$ by (4.111 instead of $T = (1/2)m\,v^2$.

There is, however, a possible new pitfall. As always, the Lagrangian $T - U$ is to be written from the perspective of an *inertial* observer. In relativistic settings, the correct transformations linking inertial observers are the Lorentz transformations. This implies that our Lagrangian should now be invariant under Lorentz transformations, not Galilean. The kinetic energy term (4.111)) is indeed Lorentz invariant since it arises from the integral

over proper time (see equation (3.67)). We thus *have* to make sure that the potential energy term U in the Lagrangian is also Lorentz invariant, independently. Not any old force law is allowed! We will tackle the transformation properties of the action in Chapter 6. And we will come back to this issue in Chapter 8 when we encounter a full Lorentz invariant force law — the electromagnetic force. In the meantime, it is worthwhile emphasizing that traditional mechanics force laws, such as Newtonian gravity, are not Lorentz invariant and hence should only be considered in approximate Newtonian settings with $L = (1/2)m\,v^2 - U$ — requiring only Galilean invariance from the action.

4.9 Summary

In this chapter we have presented a variational approach to classical mechanics, which is at the very heart of the subject. The variational approach is in fact the central theme of this book.

We began the chapter by describing a conservative mechanical system by N generalized coordinates $q_k(t)$, with $k = 1, 2, \ldots, N$, and then defining the **Lagrangian**

$$L(t, q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...) = T - U \tag{4.113}$$

as the difference between the kinetic and potential energies of the system, expressed in terms of the generalized coordinates q_k, generalized velocities $\dot{q}_k$, and time t. We then define the **action** $S[q_k(t)]$ of the system as the functional consisting of the time integral over the **Lagrangian** $L(t, q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...)$, from a starting time t_a to an ending time t_b,

$$S[q_k(t)] = \int_{t_a}^{t_b} dt\, L(t, q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...) \equiv \int_{t_a}^{t_b} dt\, L(t, q_k, \dot{q}_k)\ , \tag{4.114}$$

where it is understood that the particle or system of particles begins at some definite position $(q_1, q_2, ...)_a$ at time t_a and ends at some definite position $(q_1, q_2, ...)_b$ at time t_b.

Hamilton's principle then proposes that, for trajectories $q_k(t)$ where the action S is *stationary* — *i.e.*, when

$$\delta S = \delta \int_{t_a}^{t_b} L(t, q_k, \dot{q}_k)\ dt = 0 \tag{4.115}$$

the coordinates $q_k(t)$'s satisfy the equations of motion for the system with the prescribed boundary conditions at t_a and t_b. That is, the variational principle $\delta S = 0$ gives the **Lagrange equations**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0. \qquad (k = 1, 2, \ldots, N) \tag{4.116}$$

which are the differential equations of motion of the system.

We also defined the **Hamiltonian** of the system to be

$$H \equiv \dot{q}_k \; p_k - L \tag{4.117}$$

where a sum over k is implied, and the generalized momenta p_k are defined to be

$$p_k = \frac{\partial L}{\partial \dot{q}_k}. \tag{4.118}$$

Then

$$\frac{\partial L}{\partial t} = -\frac{dH}{dt}, \tag{4.119}$$

so if L is not an explicit function of time, it follows that the Hamiltonian is conserved.

One of the strengths of the Lagrangian formalism is that it prescribes a quite straightforward algorithmic process to solve a problem. Here are the typical steps:

1. Identify the degrees of freedom of each particle or object consistent with any constraints, and choose an appropriate set of generalized coordinates q_k for each.

2. Write the square of the velocity for each particle in terms of any convenient coordinates, usually Cartesian coordinates, in *some inertial reference frame.* Then reexpress the kinetic energy in terms of the generalized coordinates q_k, the generalized velocities $\dot{q}_k$, and the time (if needed); *i.e.*, , $v^2 = v^2(q_k, \dot{q}_k, t)$. Then write the *total* kinetic energy T in terms of these v's.

3. Write the *total* potential energy in the form $U = U(q_k, t)$.

4. Write the Lagrangian $L(t, q_1, q_2, ..., \dot{q}_1, \dot{q}_2, ...) = T - U$

5. Identify any cyclic coordinates in L; that is, identify any coordinate q_l missing in the Lagrangian, even though its corresponding generalized velocity $\dot{q}_l$ is present. In this case the corresponding generalized momentum $p_l \equiv \partial L/\partial \dot{q}_l$ is *conserved*. This gives a highly-valued first integral of motion, *i.e.*, , a differential equation that is first order rather than second order.

6. If L is not an explicit function of time, then the Hamiltonian $H \equiv \dot{q}_k\ p_k - L$ is conserved, providing another first integral of motion. Here the generalized momentum $p_i = \partial L/\partial \dot{q}_i$.

7. If there are more generalized coordinates in the problem than first integrals identified in the preceding steps, then one or more of the Lagrange equations of motion

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0. \tag{4.120}$$

   must be used as well, to obtain a complete set of differential equations. That is, if there are N generalized coordinates, we will generally need N mutually independent differential equations whose solutions will give the coordinates as functions of time. Some of these may be first-order equations, each corresponding to a conserved quantity, while others may be second-order equations.

If the problem is simple, exact analytic solutions of the equations may be possible. If not, we can always solve the equations numerically on a computer. There is also a very common intermediate situation, when one or more of the equations is too difficult to solve exactly, but approximate analytic techniques can be used to find the answer as accurately as required. Finding a sufficiently accurate approximate analytic technique to solve a particular problem can be very entertaining, requiring as much creativity as setting up the physical problem in the first place. One of the pleasures of doing physics is to find a way, no holds barred, to solve a problem to the extent we need it solved, using back-of-the-envelope calculations, dimensional reasoning, approximate analytical techniques, numerical calculations, or whatever it takes to get the job done.

Problems

PROBLEM 4-1: Example 2 featured a bead sliding on a vertically-oriented helix of radius R. The angle θ about the symmetry axis was related to its vertical coordinate z on the wire by $\theta = \alpha z$. There is a uniform gravitational field g vertically downward. (a) Rewrite the Lagrangian and find the Lagrange equation, using z as the generalized coordinate. (b) Are there any conserved dynamical quantities? (c) Write the simplest differential equation of motion of the bead, and go as far as you can to solve analytically for the z as a function of time.

PROBLEM 4-2: In Example 3 we found the equation of motion of a block on an inclined plane, using the generalized coordinate s, the distance of the block from the bottom of the incline. Solve the equation for $s(t)$, in terms of an arbitrary initial position $s(0)$ and velocity $\dot{s}(0)$.

PROBLEM 4-3: A bead of mass m is placed on a vertically-oriented circular hoop of radius R and negligible mass that is free to rotate about a vertical axis through its center. There is a uniform downward-directed gravitational field g. (a) Using polar and azimuthal angles θ and φ as generalized coordinates, find the kinetic and potential energies of the bead. (b) Find the equations of motion using Lagrange's equations. (c) Write down two first integrals of the motion. What is the physical significance of each?

PROBLEM 4-4: A particle of mass m slides inside a smooth hemispherical bowl of radius R. Beginning with spherical coordinates r, θ and φ to describe the dynamics, select generalized coordinates, write the Lagrangian, and find the differential equations of motion of the particle.

PROBLEM 4-5: A particle moves in a cylindrically symmetric potential $V(r, z)$. Use cylindrical coordinates r, θ, and z to parameterize the space.

(a) Write the Lagrangian for an unconstrained particle of mass m (using cylindrical coordinates) in the presence of this potential.

(b) Write the Lagrange equations of motion for r, θ and z.

(c) Identify and cyclic coordinates, and write a first integral corresponding to each.

PROBLEM 4-6: A spring pendulum has the pendulum bob of mass m attached to a spring of force-constant k and unstretched length R_0. The pendulum is constrained to swing in a vertical plane (*i.e.*, two degrees of freedoms are at work).

(a) Write the Lagrangian for the blob. Do not forget the potential for the spring.

(b) Write the equations of motion for θ and r. Stare at the resulting differential equations and write me a brief thank you note for not asking you to solve them.

PROBLEM 4-7: A particle of mass m slides inside a smooth paraboloid of revolution whose axis of symmetry z is vertical. The surface is defined by the equation $z = \alpha r^2$, where z and r are cylindrical coordinates, and α is a constant. There is a uniform gravitational field g. (a) Select two generalized coordinates for m. (b) Find T, U, and L. (c) Identify any ignorable

coordinates, and any conserved quantities. (d) Show that there are two first integrals of motion, and find the corresponding equations.

PROBLEM 4-8: Repeat the preceding problem for a particle sliding inside a smooth cone defined by $z = \alpha r$.

PROBLEM 4-9: A spring pendulum features a pendulum bob of mass m attached to one end of a spring of force-constant k and unstretched length R. The other end of the spring is attached to a fixed point on the ceiling. The pendulum is allowed to swing in a plane. Use r, the distance of the bob from the fixed point, and θ, the angle of the spring relative to the vertical, as generalized coordinates. (a) Find the kinetic and potential energies of the bob in terms of the generalized coordinates and velocities. (b) Find the Lagrangian. Are there any ignorable coordinates? (c) Are there any conserved quantities? (d) Find a complete set of equations of motion, including as many first integrals as possible.

PROBLEM 4-10: A pendulum is constructed from a bob of mass m on one end of a light string of length D. The other end of string is attached to the top of a circular cylinder of radius R ($R < 2D/\pi$). The string makes an angle θ with the vertical, as shown below, as the pendulum swings in the plane. There is a uniform gravity g directed downward. (a) Find the Lagrangian and write out Lagrange's equation using θ as the generalized coordinate. (b) Identify any first integrals of motion. (c) Find the frequency of small oscillations about the stable equilibrium point.

PROBLEM 4-11: A particle moves with a cylindrically symmetric potential energy $U = U(r, z)$ where r, θ, z are cylindrical coordinates. (a) Write the Lagrangian for an unconstrained particle of mass m in this case. (b) Are there any cyclic coordinates? If so, what symmetries do they correspond to, and what are the resulting constants of the motion? (c) Write the Lagrange equation for each cyclic coordinate. (d) Find the Hamiltonian H. Is it conserved? (e) Find the total energy E. Is $E = H$? is E conserved? (f) Write the simplest (*i.e.*, , lowest-order) complete set of differential equations of motion of the particle.

PROBLEM 4-12: A plane pendulum is made with a plumb bob of mass m hanging on a Hooke's-law spring of negligible mass, force constant k, and unstretched length ℓ_0. The spring can stretch but is not allowed to bend. There is a uniform downward gravitational field g. (a) Select generalized coordinates for the bob, and find the Lagrangian in terms of them. (b) Write out the Lagrange equations of motion (c) Are there any conserved quantities? If so, write down the corresponding conservation law(s). (d) If the bob is swinging back and forth, find the frequency of small oscillations in the general case where the spring can change its length while the bob is swinging back and forth.

PROBLEM 4-13: Motion in a slowly-changing uniform electric field

A particle of mass m and charge q moves within a parallel-plate capacitor whose charge Q is decays exponentially with time, $Q = Q_0 e^{-t/\tau}$, where τ is the time constant of the decay. Find the equations of motion of the particle.

PROBLEM 4-14: A particle of mass m travels between two points $x = 0$ and $x = x_1$ on Earth's surface, leaving at time $t = 0$ and arriving at $t = t_1$. The gravitational field g is uniform. (a) Suppose m moves along the ground (keeping altitude $z = 0$) at steady speed. Find the total action S to go by this path. (b) Suppose instead that m moves along the least-action parabolic path. Show that the action in this case is

$$S = \frac{mx_1^2}{2t_1} - \frac{mg^2t_1^3}{24} \tag{4.121}$$

and verify that it is less than the action for the straight-line path of part (a).

PROBLEM 4-15: Suppose the particle of the preceding problem moves instead at constant speed along a isoceles triangular path between the beginning point and the end point, with the high point at height z_1 above the ground, at $x = x_1/2$ and $t = t_1/2$. (a) Find the action for this path. (b) Find the altitude z_1 corresponding to the least-action path among this class of constant-speed triangular paths. (c) Verify that the total action for this path is greater than that of the parabolic path of the preceding problem.

PROBLEM 4-16: A plane pendulum consists of a light rod of length R supporting a plumb bob of mass m in a uniform gravitational field g. The point of support of the top end of the rod is forced to oscillate back and forth in the horizontal direction with $x = A\cos\omega t$. Using the angle θ of the bob from the vertical as the generalized coordinate, (a) find the Lagrangian of the plumb bob. (b) Are there any conserved dynamical quantities? (c) Find the simplest differential equation of motion of the bob.

PROBLEM 4-17: Solve the preceding problem if instead of being forced to oscillate in the horizontal direction, the upper end of the rod is forced to oscillate in the vertical direction with $y = A\cos\omega t$.

PROBLEM 4-18: A particle of mass m on a frictionless table top is attached to one end of a light string. The other end of the string is threaded through a small hole in the table top, and held by a person under the table. If given a sideways velocity v_0, the particle circles the hole with radius r_0. At time $t = 0$ the mass reaches an angle defined to be $\theta = 0$ on the table top, and the person under the table pulls on the string so that the length of the string above the table becomes $r(t) = r_0 - \alpha t$ for a period of time thereafter, where α is a constant. Using θ as the generalized coordinate of the particle, find its Lagrangian, identify any conserved quantities, finds its simplest differential equation of motion, and get as far as you can using analytic means alone toward finding the solution $\theta(t)$ (or $t(\theta)$).

PROBLEM 4-19: A rod is bent in the middle by angle α. The bottom portion is kept vertical and the top portion is therefore oriented at angle α to the vertical. A bead of mass m is slipped onto the top portion and the bottom portion is forced by a motor to rotate at constant angular speed ω about the vertical axis. (a) Define a generalized coordinate for the bead and write down the Lagrangian. (b) Identify any conserved quantity or quantitiies and explain why it (or they) are conserved. (c) Find the generalized momentum of the bead and the Hamiltonian. (d) Are there any equilibrium points of the bead? If so, are they stable or unstable?

PROBLEM 4-20: A wire is bent into the shape of a cycloid, defined by the parametric equations $x = A(\varphi + \sin\varphi)$ and $y = A(1 - \cos\varphi)$, where φ is the parameter ($-\pi < \varphi < \pi$), and A is a constant. The wire is in a vertical plane, and is spun at constant angular velocity ω about a vertical axis through its center. A bead of mass m is slipped onto the wire. (a) Find the Lagrangian of the bead, using the parameter φ as the generalized coordinate. (b) Identify any first integral of motion of the bead. (c) Are there any equilibrium points of the bead? Are they stable or unstable? For any stable equilibrium point, find the frequency of small oscillations about that point.

PROBLEM 4-21: *Center of mass and relative coordinates.* Show that for two particles moving in one dimension, with coordinates x_1 and x_2, with a potential that depends only upon their separation $x_2 - x_1$, then the Lagrangian

$$L = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - U(x_2 - x_1) \tag{4.122}$$

can be rewritten in the form

$$L = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}\mu\dot{x}'^2 - U(x'), \tag{4.123}$$

where $M = m_1 + m_2$ is the total mass and $\mu = m_1m_2/M$ is the "reduced mass" of the system, and $X = (m_1x_1 + m_2x_2)/M$ is the center of mass coordinate and $x' = x_2 - x_1$ is the relative coordinate.

PROBLEM 4-22: Two blocks of equal mass m, connected by a Hooke's-law spring of unstretched length ℓ, are free to move in one dimension. Find the equations of motion of the system, using the relative and center of mass coordinates introduced in the preceding problem.

PROBLEM 4-23: A small block of mass m and a weight of mass M are connected by a string of length D. The string has been threaded through a small hole in a tabletop, so the block can slide without friction on the tabletop, while the weight hangs vertically beneath the tabletop. We can let the hole be the origin of coordinates, and use polar coordinates r, θ for the block, where r is the block's distance from the hole, and z for the distance of the weight below the tabletop. (a) Using generalized coordinates r and θ, write down the Lagrangian of the system of block plus weight. (b) Write down a complete set of first integrals of the motion, explaining the physical meaning of each. (c) Show that the first integrals can be combined to give an equation of the form

$$E = \frac{1}{2}(M + m)\dot{r}^2 + U_{\text{eff}}(r) \tag{4.124}$$

and write out an expression for $U_{\text{eff}}(r)$. (d) Find the radius of a circular orbit of the block in terms of constants of the motion. (e) Now suppose the block executes small oscillations about a circular orbit. What is the frequency of these oscillations? Is the resulting orbit of the block open or closed? That is, does the perturbed orbit of the block continually return to its former position or not?

PROBLEM 4-24: The Moon has no atmosphere, so it would be possible in principle to shoot projectiles off its surface with the escape velocity or even higher. The projectiles might be mined Moon material, shot into space to use as raw material for building structures there. One way to raise material to the escape velocity would be to construct a huge boom that would continually swing around in a horizontal plane with constant angular velocity ω. Buckets of material would be dropped onto the boom at some small distance r_0 from the rotation axis, with no initial radial velocity. The buckets would then be thrown outward by the boom's rotation and would come off the end of the boom with the escape velocity if the boom is long enough and if ω is large enough. (a) Find the Lagrangian for a bucket of material of mass m. (b) Find its equation of motion. (c) Solve it for $r(t)$, subject to the given initial conditions (*i.e.*, , be sure that $r = r_0$ and $\dot{r} = 0$ at $t = 0$.) (d) If the boom has radius R, find the radial and tangential components of the bucket's velocity, and its total speed, as it emerges from the end of the boom. (e) Find the power input $P = dE/dt$ into a bucket of mass m as a function of time. Is the power input larger at the beginning or end of the bucket's journey along the boom? (f) Find the torque exerted by the boom on the bucket, as a function of the position r of the bucket on the boom. There would be an equal but opposite torque back on the boom, caused by the bucket, which might break the boom. At what part of the bucket's journey would this torque most likely break the boom? (g) If $R = 100$ meters and $r_0 = 1$ meter, what must be the rotational period of the boom so that buckets will reach the Moon's escape speed as they fly off the boom?

PROBLEM 4-25: Consider a vertical circular hoop of radius R rotating about a vertical axis with constant angular velocity Ω. A bead of mass m is threaded on the hoop. Denote the angle along the hoop of the bead as measured from the vertical as θ.

(a) Write the Lagrangian and equations of motion.

(b) For small angles $\theta \ll 1$ (in radians), we have the approximate expressions $\cos\theta \sim 1$ and $\sin\theta \sim \theta$. In this regime, simplify the equation of motion for θ. Say a sentence about the solution $\theta(t)$ that you expect.

(c) In the result of part (a), rewrite things in terms of the new coordinate $\alpha \equiv \theta - \pi$. Note the mildly exciting identities $\cos(\alpha + \pi) = -\cos\alpha$ and $\sin(\alpha + \pi) = -\sin\alpha$. Now consider the regime $\alpha \sim 0$ and simplify the equation of motion for α.

PROBLEM 4-26: Consider the Lagrangian $L' = m\,\dot{x}\,\dot{y} - k\,x\,y$ for a particle free to move in two dimensions, where x and y are Cartesian coordinates, and m and k are constants.

(a) Find the equations of motion for the system.

(b) Confirm that the answer to part (a) is the same if we were to use instead the Lagrangian for the harmonic oscillator $L = (1/2)m(\dot{x}^2 + \dot{y}^2) - (1/2)k(x^2 + y^2)$.

(c) Show that L and L' do *not* differ by a total derivative!

PROBLEM 4-27: In certain situations, it is possible to incorporate frictional effects in a simple way into a Lagrangian problem. As an example, consider the Lagrangian

$$L = e^{\gamma t}\left(\frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2\right)\ . \tag{4.125}$$

(a) Find the equation of motion for the system.
(b) Do a coordinate change $s = e^{\gamma t/2} q$. Rewrite the dynamics in terms of s.
(c) How would you describe the system?

PROBLEM 4-28: Consider a particle moving in three dimensions with Lagrangian $L = (1/2)m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + a\dot{x} = b$, where a and b are constants. (a) Find the equations of motion and show that the particle moves in a straight line at constant speed, so that it must be a free particle. (b) The result of (a) shows that there must be another reference frame (x', y', z') such that the Lagrangian is just the usual free-particle Lagrangian $L' = (1/2)m(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2)$. However, L' may also be allowed an additive constant, which cannot show up in Lagrange's equations. Find the Galilean transformation between (x, y, z) and (x', y', z') and find the velocity of the new primed frame in terms of a and b.

PROBLEM 4-29: Consider a Lagrangian $L' = L + df/dt$, where the Lagrangian is $L = L(t, q_k, \dot{q}_k)$, and the function $f = f(q_k, t)$. (a) Show that $L' = L'(q_k, \dot{q}_k, t)$, so that it depends upon the proper variables. Show that this would not generally be true if f were allowed to depend upon the $\dot{q}'_k s$. (b) Show that L' obeys Lagrange's equations if L does, by substituting L' into Lagrange's equations. Therefore the equations of motion are the same using L' as using L, so the Lagrangian of a particle is not unique. (This problem requires care in taking total and partial derivatives!)

PROBLEM 4-30: Show that the function L' given in the preceding problem must obey Lagrange's equations if L does, directly from the principle of stationary action. Lagrange's equations do not have to be written down for this proof!

PROBLEM 4-31: Consider the Lagrangian $L' = m\dot{x}\dot{y} - kxy$ for a particle free to move in two dimensions, where x and y are Cartesian coordinates, and m and k are constants. (a) Show that his Lagrangian gives the equations of motion appropriate for a two-dimension simple harmonic oscillator. Therefore as far as the motion of the particle is concerned, L' is equivalent to $L = (1/2)m(\dot{x}^2 + \dot{x}^2) - (1/2)k(x^2 + y^2)$. (b) Show that L' and L do *not* differ by the total time derivative of any function $f(x, y)$. Therefore L' is not a member of the class of Lagrangians mentioned in the preceding problems, so there are even more Lagrangians describing a particle than suggested before.

PROBLEM 4-32: Consider a Lagrangian that depends on second derivatives of the coordinates

$$L = L(q_k, \dot{q}_k, \ddot{q}_k, t) \ . \tag{4.126}$$

Through the variational principle, show that the resulting differential equations of motion are third order in the time derivative.

PROBLEM 4-33: A pendulum consists of a plumb bob of mass m on the end of a string that swings back and forth in a plane. The upper end of the string passes through a small hole in the ceiling, and the angle θ of the bob relative to the vertical changes with time as it swings back and forth. The string is pulled upward at constant rate through the hole, so the length

R of the pendulum decreases at a constant rate, with $dR/dt = -\alpha$. (a) Find the Lagrangian of the bob, using θ as the generalized coordinate. (b) Find the Hamiltonian H. Is H equal to the energy E? Why or why not? (c) Is either H or E conserved? Why or why not?

PROBLEM 4-34: A spherical pendulum consists of a particle of mass m on the end of a string of length R. The position of the particle can be described by a polar angle θ and an azimuthal angle φ. The length of the string decreases at the rate $dR/dt = -f(t)$, where $f(t)$ is a positive function of time. (a) Find the Lagrangian of the particle, using θ and φ as generalized coordinates. (b) Find the Hamiltonian H. Is H equal to the energy? Why of why not? (c) Is either E or H conserved? Why or why not?

PROBLEM 4-35: Equation 4.100 for the Hamiltonian of a bead on a parabolic wire turning with constant angular velocity ω is

$$H = \frac{1}{2}m[(1 + 4\alpha^2 r^2)\dot{r}^2 - r^2\omega^2] + mg\alpha r^2, \tag{4.127}$$

where H is a constant. Reduce the problem to quadrature: That is, find an equation for the time t is terms of an integral over r.

PROBLEM 4-36: A bead of mass m is placed on a vertically-oriented circular hoop of radius R that is forced to rotate with constant angular velocity ω about a vertical axis through its center. (a) Using the polar angle θ measured up from the bottom as the single generalized coordinate, find the kinetic and potential energies of the bead. (Remember that the bead has motion due to the forced rotation of the hoop as well as motion due to changing θ.) (b) Find the bead's equation of motion using Lagrange's equation. (c) Is its energy conserved? Why or why not? (c) Find its Hamiltonian. Is H conserved? Why or why not? (d) Is $E = H$? Why or why not? (e) Find the equilibrium angle θ_0 for the bead as a function of the hoop's angular velocity ω. Sketch a graph of θ_0 versus ω. Notice that there is a "phase transition" at a certain critical velocity $\omega_{\rm crit}$. (b) Find the frequency of small oscillations of the bead about the equilibrium angle θ_0, as a function of ω.

PROBLEM 4-37: A bead of mass m is placed on a vertically-oriented elliptical hoop that is forced to rotate with constant angular velocity ω about a vertical axis through its center. The ellipse is defined by $(x/a)^2 + (y/b)^2 = 1$ where a and b are the semimajor and semiminor axes of the ellipse, and suppose that the vertical axis is the semiminor axis. (a) Choose a generalized coordinate for the bead and find the Lagrangian (b) Is the bead's energy conserved? Why or why not? (c) Is the bead's angular momentum conserved about the vertical axis? Why or why not? (d) Find the bead's Hamiltonian. Is H conserved? Why or why not? (d) Is $E = H$? Why or why not? (e) Given ω, is there an equilibrium position of the bead, and is it stable or unstable?

PROBLEM 4-38: In Example 9 we analyzed the case of a bead on a rotating parabolic wire. The energy of the bead was not conserved, but the Hamiltonian was:

$$H = \frac{1}{2}m(1 + 4\alpha^2 r^2)\dot{r}^2 + \text{“}U''_{\rm eff} = \text{constant},$$

where

$$\text{“}U''_{\text{eff}} = \frac{1}{2}mr^2(2g\alpha^2 - \omega^2).$$

There is an equilibrium point at $r = 0$ which is unstable if $\omega > \omega_0 \equiv \sqrt{2g}\ \alpha$, neutrally stable if $\omega = \omega_0$, and stable if $\omega < \omega_0$. Find the frequency of small oscillations about $r = 0$ if $\omega < \omega_0$. *Hint*: Note that for very small oscillations near $r = 0$ the quantity $(1 + 4\alpha^2 r^2) \cong 1$ to an excellent approximation, so the Hamiltonian is

$$H \cong \frac{1}{2}m\dot{r}^2 + \frac{1}{2}kr^2$$

where the constant $k = m(2g\alpha^2 - \omega^2)$.

Appendix A

When is $H \neq E$?

In Example 9 the Hamiltonian H was not equal to E. Why were they different, and why was H conserved while E was not?

The definition $H = \dot{q}_k \partial L/\partial \dot{q}_k - L$ (using the Einstein summation convention, implying a sum over k in the first term on the right since k is repeated in that term) contains the Lagrangian $L = T - U$, where only the kinetic energy T depends upon the generalized velocities $\dot{q}_k$; therefore

$$H = \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} + U - T. \tag{A.1}$$

Now let $\boldsymbol{r}(q_k, t)$ be the position vector of the particle from some arbitrary origin fixed in an inertial frame, in terms of the time and any or all of the generalized coordinates q_k. Then the velocity of the particle is

$$\boldsymbol{v} = \frac{d\boldsymbol{r}(q_k, t)}{dt} = \frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial q_l}\dot{q}_l, \tag{A.2}$$

because $\boldsymbol{r}$ can change with time either from an explicit time dependence or because one or more of the generalized coordinates changes with time. Therefore its kinetic energy is $T = (1/2)mv^2 = (1/2)m\boldsymbol{v} \cdot \boldsymbol{v}$, which is

$$T = \frac{1}{2}\left[m\frac{\partial \boldsymbol{r}}{\partial t} \cdot \frac{\partial \boldsymbol{r}}{\partial t} + 2\ \frac{\partial \boldsymbol{r}}{\partial t} \cdot \frac{\partial \boldsymbol{r}}{\partial q_l}\dot{q}_l + \frac{\partial \boldsymbol{r}}{\partial q_l}\dot{q}_l \cdot \frac{\partial \boldsymbol{r}}{\partial q_m}\dot{q}_m\right] \tag{A.3}$$

where we have used a different dummy summation index m in the final factor to distinguish it from the sum over l is the preceding factor. That is, by the Einstein summation convention the final term above is actually the product of

two sums, one over l and one over m. Now we can take the partial derivative of T with respect to a particular one of the generalized velocities $\dot{q}_k$,

$$\frac{\partial T}{\partial \dot{q}_k} = \frac{1}{2} m \left[2 \, \frac{\partial \boldsymbol{r}}{\partial t} \cdot \frac{\partial \boldsymbol{r}}{\partial q_k} + 2 \, \frac{\partial \boldsymbol{r}}{\partial q_k} \cdot \frac{\partial \boldsymbol{r}}{\partial q_l} \dot{q}_l \right] \tag{A.4}$$

where there is a factor of two in the second term because $\dot{q}_k$ occurs in both summations in the last term of the expression for T. Therefore the sum

$$\begin{aligned} \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} &= m \left[\frac{\partial \boldsymbol{r}}{\partial t} \cdot \frac{\partial \boldsymbol{r}}{\partial q_k} \dot{q}_k + \frac{\partial \boldsymbol{r}}{\partial q_k} \dot{q}_k \cdot \frac{\partial \boldsymbol{r}}{\partial q_l} \dot{q}_l \right] \\ &= 2T - m \frac{\partial \boldsymbol{r}}{\partial t} \cdot \left[\frac{\partial \boldsymbol{r}}{\partial t} + \frac{\partial \boldsymbol{r}}{\partial q_k} \dot{q}_k \right] \\ &= 2T - m \frac{\partial \boldsymbol{r}}{\partial t} \cdot \frac{d\boldsymbol{r}}{dt}, \end{aligned} \tag{A.5}$$

using equations (A.3)) and (A.4. The Hamiltonian H can therefore be written

$$H = E - m \frac{d\boldsymbol{r}}{dt} \cdot \frac{\partial \boldsymbol{r}}{\partial t} = E - \boldsymbol{p} \cdot \frac{\partial \boldsymbol{r}}{\partial t} \tag{A.6}$$

where $\boldsymbol{p}$ is the momentum of the particle in the chosen inertial frame. If the transformation $\boldsymbol{r} = \boldsymbol{r}(q_k, t)$ between the Cartesian coordinates $\boldsymbol{r} = (x, y, z)$ and the generalized coordinates q_k happens not to be an explicit function of time, *i.e.*, , if $\partial \boldsymbol{r}/\partial t = 0$, then the Hamiltonian is just $T + U$. This case occurs when there are no constraints or when any constraints are fixed in space. But if the constraints are moving, then the transformation $\boldsymbol{r} = \boldsymbol{r}(q_k, t)$ does generally depend upon time, and so in the likely case that there is a component of $\partial \boldsymbol{r}/\partial t$ in the direction of the particle's momentum $\boldsymbol{p}$, the Hamiltonian is not equal to $T + U$.

For the problem of the bead on a rotating parabolic wire, where the constraint is obviously moving, the position vector of the bead can be taken to be $\boldsymbol{r} = (x, y, z) = (r \cos \omega t, r \sin \omega t, \alpha r^2)$. In that case we found that $H = T + U - m\omega^2 r^2$, and it is easy to show that $m\omega^2 r^2 = \boldsymbol{p} \cdot \partial \boldsymbol{r}/\partial t$, as required by equation (A.6)). It is clear that E is not conserved in this case because the rotating wire is continually doing work on the bead. The wire exerts a force F^θ in the tangential direction, which causes work to be done at the rate $dW/dt = F^\theta v^\theta = F^\theta r \omega$. From the elementary relationship $N^z = dL^z/dt$, with torque $N^z = F^\theta r$ and angular momentum $L^z = (\boldsymbol{r} \times \boldsymbol{p}^z = mr^2\omega$, it follows that

$$\frac{dW}{dt} = \omega \frac{d(mr^2\omega)}{dt} = m\omega^2 \frac{dr^2}{dt}, \tag{A.7}$$

so that the work done by the wire is $W = m\omega^2 r^2$ plus a constant of integration, which depends upon the initial location of the bead. Thus the energy' $E = T + U$ of the bead increases by the work done upon it by the wire, so that E *minus* the work done must be conserved, and that difference $E - m\omega^2 r^2 = H$.

Note that:

1. H is conserved if the Lagrangian L is not an explicit function of time.

2. $H = E$ if the coordinate transformation $\boldsymbol{r} = \boldsymbol{r}(q_k, t)$ is not an explicit function of time. Therefore it is possible to have $E = H$, with both E and H conserved, or neither conserved, and it is also possible to have $E \neq H$, with both conserved, neither conserved, or only one of the two conserved. Examples are given in the problems.