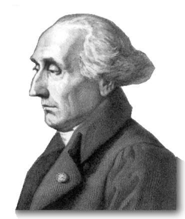
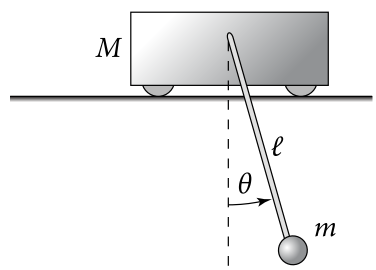
Problem Set 7 Solution

Friday, 4 November 2011



For each problem, please account for all ND degrees of freedom, where N is the number of particles and D the number of dimensions.

Problem 1 – Really practical! A cart of mass M glides without friction on a smooth horizontal surface. Suspended from the cart at a frictionless pivot by a massless rigid rod of length ℓ is a mass m, which is free to move in the plane. Find the eigenfrequencies and describe the corresponding normal modes.



Solution: There is no force on the system in the horizontal (x) direction, so horizontal momentum is conserved. Because the surface is flat and horizontal, there is no restoring force on the cart should it be set in motion at uniform horizontal velocity along with the pendulum. One "normal mode," therefore, consists in translation of the center of mass at steady speed. It has zero frequency: it never comes back. If you think this is cheap, imagine what should happen if the cart rolls in a very weakly curved bowl. It could roll up the side of the bowl a little bit, turn around, and slide back down. This could take as long as we like, if we make the bowl shallower and shallower. In the limit that the bowl goes flat, the period goes to infinity.

The other normal mode has the cart moving left when the bob moves right, keeping the center of mass fixed. Let x be the position of the cart with respect to the equilibrium position. The position of the mass is then

$$\mathbf{r} = (x + \ell \sin\theta)\hat{\mathbf{x}} + \ell\cos\theta\hat{\mathbf{y}} \tag{1}$$

The position of the center of mass in the horizontal direction is

$$m(x + \ell\sin\theta) + Mx = 0 \qquad \Longrightarrow \qquad x = -\frac{m\ell\sin\theta}{m+M} \tag{2}$$

The kinetic energy of the cart is therefore

$$T_{\text{cart}} = \frac{M}{2}\dot{x}^2 = \frac{M}{2}\left(-\frac{m\ell\dot{\theta}\cos\theta}{m+M}\right)^2$$

Now θ is a small quantity, and so is $\dot{\theta}$. Since we are expanding only to quadratic order, we may approximate $\cos\theta \approx 1$, since there are already two factors of the small quantity θ from the $\dot{\theta}^2$ term. Therefore,

$$T_{\text{cart}} = \frac{m^2 M \ell^2 \dot{\theta}^2}{2(m+M)^2}$$

We can calculate the kinetic energy of the pendulum using its position in Cartesians given by Eq. (1). But first, eliminate x using Eq. (2):

$$\mathbf{r} = \ell\sin\theta\left(1 - \frac{m}{m+M}\right)\hat{\mathbf{x}} + \ell\cos\theta\hat{\mathbf{y}} = \ell\sin\theta\frac{M}{m+M}\hat{\mathbf{x}} + \ell\cos\theta\hat{\mathbf{y}}$$
$$\mathbf{v} = \ell\dot{\theta}\cos\theta\left(\frac{M}{m+M}\right)\hat{\mathbf{x}} - \ell\dot{\theta}\sin\theta\hat{\mathbf{y}}$$

The kinetic energy of the pendulum is therefore

$$T_{\text{pend}} = \frac{m}{2}\ell^2\dot{\theta}^2\left(\frac{M}{m+M}\right)^2$$

where we have dropped all terms higher order than $\dot{\theta}^2$. The total kinetic energy is thus

$$T = \frac{mM\ell^2\dot{\theta}^2}{2}\frac{m+M}{(m+M)^2} = \frac{1}{2}\left(\frac{mM\ell^2}{m+M}\right)\dot{\theta}^2 \tag{3}$$

The potential energy is

$$V = mg\ell(1-\cos\theta) \approx \frac{m}{2}g\ell\theta^2$$

where we approximate through second order in θ. The total energy of the system is thus

$$E = T + V = \frac{1}{2}\left(\frac{mM\ell^2}{m+M}\right)\dot{\theta}^2 + \frac{1}{2}(mg\ell)\,\theta^2$$

The frequency of small oscillations is therefore

$$\omega = \sqrt{\frac{mg\ell}{\frac{mM\ell^2}{m+M}}} = \sqrt{\frac{g(m+M)}{\ell M}} = \sqrt{\frac{g}{\ell}\left(1+\frac{m}{M}\right)}$$

Does this make sense? Suppose that $M \gg m$. The cart just sits there and the pendulum swings, so its frequency should be $\sqrt{g/\ell}$. That checks. As mentioned previously, this mode corresponds to motion of the bob and cart in opposition.

Problem 2 A thin hoop of radius R and mass M oscillates in its own plane with one point of the hoop fixed. Attached to the hoop is a small mass m constrained to move (without friction) along the hoop. Consider only small oscillations.

(a) Show that the eigenfrequencies are

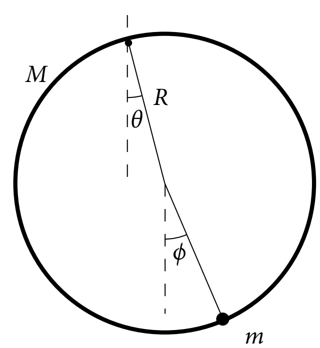
$$\omega_1 = \sqrt{\left(1 + \frac{m}{M}\right)\frac{g}{R}} \qquad \text{and} \qquad \omega_2 = \sqrt{\frac{g}{2R}}$$

(b) Describe the normal mode corresponding to each eigenfrequency.

(c) Find the initial conditions that produce motion in each normal mode without exciting the other.

Solution:

As discussed in the notes, our strategy will be to find the mass and spring matrices, $\mathbb{M}$ and $\mathbb{A}$, based on the expressions for kinetic and potential energy for the system. Let the angular displacement of the hoop be θ, so that its potential energy is



$$U_{\text{hoop}} = MgR(1 - \cos\theta) \approx MgR\frac{\theta^2}{2}$$

when expanded to second order in θ, since the center of mass of the hoop is at R from the pivot. The moment of inertia of the hoop about its center is MR^2, as all the mass is equidistant from the center. Using the parallel axis theorem, we find that the kinetic energy for rotation about a point on the perimeter is

$$T_{\text{hoop}} = \frac{1}{2}(MR^2 + MR^2)\dot{\theta}^2 = MR^2\dot{\theta}^2$$

Let ϕ represent the angular position of the bead with respect to the vertical. Then the bead's motion is like that of the double pendulum, with

$$U_{\text{bead}} = mgR(2 - \cos\theta - \cos\phi) \approx \frac{mgR}{2}(\theta^2 + \phi^2)$$

$$T_{\text{bead}} = \frac{m}{2}R^2\left[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)\right] \approx \frac{mR^2}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right)$$

Combining the approximations for bead and hoop gives

$$U = \frac{MgR}{2}\left(\theta^2(1 + \mu) + \mu\phi^2\right)$$

$$T = \frac{MR^2}{2}\left[\dot{\theta}^2(2 + \mu) + 2\mu\dot{\theta}\dot{\phi} + \mu\dot{\phi}^2\right]$$

where I have defined $\mu \equiv m/M$. We can now identify the two matrices,

$$\mathbb{A} = MgR\begin{pmatrix} 1+\mu & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{and} \qquad \mathbb{M} = MR^2\begin{pmatrix} 2+\mu & \mu \\ \mu & \mu \end{pmatrix}$$

The normal modes satisfy $\det(\mathbb{A} - \omega^2\mathbb{M}) = 0$. Define $\omega_0^2 = g/R$ and let $x = \omega^2/\omega_0^2$. Then we must have

$$\begin{vmatrix} 1+\mu-(2+\mu)x & -\mu x \\ -\mu x & \mu - \mu x \end{vmatrix} = 0$$

or

$$[1+\mu-(2+\mu)x](1-x)\mu - \mu^2 x^2 = 0$$

Discarding a common factor of μ, we get the quadratic equation

$$2x^2 - x(3+2\mu) + (1+\mu) = 0 \qquad \Longrightarrow \qquad x = \frac{3+2\mu \pm (1+2\mu)}{4}$$

So, the eigenvalues of x are $x_+ = 1+\mu$ and $x_- = 1/2$, meaning that the eigenfrequencies are

$$\omega_- = \frac{\omega_0}{\sqrt{2}} = \sqrt{\frac{g}{2R}} \qquad \text{and} \qquad \omega_+ = \omega_0\sqrt{1+m/M} = \sqrt{\frac{g}{R}\left(1+\frac{m}{M}\right)}$$

(b) We should anticipate normal modes in which the bead and hoop swing in the same sense, which should have lower frequency because they function like a long pendulum, and in which they move in opposition. To confirm this intuition, substitute the eigenvalues in turn into the matrix $\mathbb{A} - \omega^2\mathbb{M}$ and solve for the eigenvector:

$$x_-: \quad \begin{pmatrix} 1+\mu-(2+\mu)/2 & -\mu/2 \\ -\mu/2 & \mu/2 \end{pmatrix}\begin{pmatrix} \theta_- \\ \phi_- \end{pmatrix} = 0 \qquad \begin{pmatrix} \mu/2 & -\mu/2 \\ -\mu/2 & \mu/2 \end{pmatrix}\begin{pmatrix} \theta_- \\ \phi_- \end{pmatrix} = 0 \qquad \mathbf{A}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x_+: \quad \begin{pmatrix} 1+\mu-(2+\mu)(1+\mu) & -\mu(1+\mu) \\ -\mu(1+\mu) & \mu[1-(1+\mu)] \end{pmatrix}\begin{pmatrix} \theta_+ \\ \phi_+ \end{pmatrix} = 0$$

$$\begin{pmatrix} (1+\mu)(-1-\mu) & -\mu(1+\mu) \\ -\mu(1+\mu) & \mu^2 \end{pmatrix}\begin{pmatrix} \theta_+ \\ \phi_+ \end{pmatrix} = 0 \qquad \mathbf{A}_+ = c\begin{pmatrix} -\mu \\ 1+\mu \end{pmatrix}$$

where c is a normalization constant. Sure enough, in the lower-frequency mode the hoop and bead move in the same direction; in the higher-frequency mode, they move in opposite directions. Furthermore, in the limit that the hoop is very massive compared to the bead ($\mu \ll 1$), the hoop is approximately fixed and the bead does all the moving in the higher-frequency mode, which then has frequency $\sqrt{\frac{g}{R}}$, appropriate for a pendulum of length R.

(c) What initial conditions produce motion in each normal mode? To stimulate the low-frequency mode, start the system at rest with $\theta = \phi$. To activate the high-frequency mode, start from rest with $\phi = -(1+\mu^{-1})\theta$. Other conditions with nonzero generalized velocities are possible, too.

Problem 3 Consider a one-dimensional model of a solid consisting of alternating atoms of two types (e.g., Na and Cl). Assume that the mass of one kind of atom is m and the other is M, and let the spring constant linking the masses be α (we will assume all the springs are identical and consider only longitudinal motion). Put the origin at the zeroth mass m, and let the equilibrium positions of the first M and second m be $a/2$ and a to the right, respectively. To avoid surface effects, assume that there are N atoms of each type, where $N \ggg 1$, and use wrap-around boundary conditions. That is, the $(N-1)$st mass M links to the $(N-1)$st mass m on its left and to the zeroth mass m on its right.

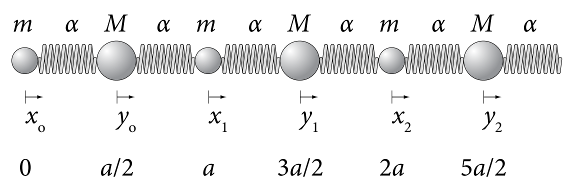
(a) Solve for the eigenfrequencies and plot their spectrum.

(b) Describe the normal modes.

(c) What is the speed of sound in the crystal?

Solution:

A portion of the very long chain is shown in the figure. Let x_j be the displacement of the jth atom of mass m from its equilibrium position at aj and y_j be the displacement of the jth atom of mass M from its equilibrium position at $a(j+1/2)$.



The equations of motion for the jth mass of each type are

$$m\ddot{x}_j = \alpha[(y_j - x_j) - (x_j - y_{j-1})] \tag{4}$$

$$M\ddot{y}_j = \alpha[(x_j - y_j) - (y_j - x_{j+1})] \tag{5}$$

We now look for traveling wave solutions (roughly) of the form $e^{i(kx-\omega t)}$, making the *Ansatz*

$$x_j = Ae^{i(kaj-\omega t)} \tag{6}$$

$$y_j = Be^{i[k(a+1/2)j-\omega t]} \tag{7}$$

Substituting into the equations of motion, and using the shorthand $\phi = kaj - \omega t$ and $\psi = k(a+1/2)j - \omega t$, we get

$$-m\omega^2 Ae^{i\phi} = \alpha(-2Ae^{i\phi} + Be^{i\phi+ika/2} + Be^{i\phi-ika/2})$$

$$m\omega^2 = 2\alpha\left(1 - \frac{B}{A}\cos ka/2\right) \tag{8}$$

$$-M\omega^2 Be^{i\psi} = \alpha(-2Be^{i\psi} + Ae^{i\psi-ika/2} + Ae^{i\psi+ika/2})$$

$$M\omega^2 = 2\alpha\left(1 - \frac{A}{B}\cos ka/2\right) \tag{9}$$

Let the mass ratio be $\mu = m/M$, the amplitude ratio be $\rho = A/B$, and divide Eq. (8) by Eq. (9) to get

$$\mu = \frac{m}{M} = \frac{1 - \rho^{-1}\cos ka/2}{1 - \rho\cos ka/2}$$

Solving for ρ yields

$$\rho = \frac{A}{B} = \frac{\mu - 1 \mp \sqrt{(\mu-1)^2 + 4\mu\cos^2 ka/2}}{2\mu\cos ka/2} \tag{10}$$

(Note that I have used $\mp$ with malice of forethought, so that the sign will come out $\pm$ in the expression for the eigenfrequencies.) Substituting this expression for ρ into either Eq. (8) or Eq. (9) allows us to solve for the eigenfrequencies

$$\boxed{\omega^2 = \frac{\alpha}{m}\left[(1+\mu) \pm \sqrt{(1-\mu)^2 + 4\mu\cos^2 ka/2}\right]} \tag{11}$$

Time for a sanity check. If $\mu = 1$, the atoms are identical and we should recover the relationship derived in class, except that now the separation between nearest neighbors is $a/2$ instead of a. If $\mu = 1$, Eq. (11) becomes

$$\omega^2 = \frac{\alpha}{m}\left(2 \pm 2\cos ka/2\right)$$

If we take the negative sign, the term in parentheses is $4\sin^2 ka/4$, which does indeed agree with our previous result.

Before plotting, let's look for the sound speed. This is the ratio of ω/k for small values of ω and k. For small values of k we can approximate the cosine term as

$$\cos^2\frac{ka}{2} \approx \left(1 - \frac{1}{2!}\frac{k^2a^2}{4} + \cdots\right)^2 \approx 1 - \frac{k^2a^2}{4}$$

where we have dropped terms beyond second order in k. Therefore the radical simplifies to

$$\sqrt{1 - 2\mu + \mu^2 + 4\mu - \mu k^2a^2} = \sqrt{(1+\mu)^2 - \mu k^2a^2}$$

Remember that k is small, so we can use the binomial approximation to approximate further that

$$\sqrt{(1+\mu)^2 - \mu k^2a^2} = (1+\mu)\sqrt{1 - \mu\left(\frac{ka}{1+\mu}\right)^2} \approx (1+\mu)\left[1 - \frac{\mu}{2}\left(\frac{ka}{1+\mu}\right)^2\right]$$

Substituting into Eq. (11) gives

$$\omega^2 = \frac{\alpha(1+\mu)}{m}\left\{1 \pm \left[1 - \frac{\mu}{2}\left(\frac{ka}{1+\mu}\right)^2\right]\right\} \tag{12}$$

If we take the positive sign, then we can neglect the term with the small k in it, getting

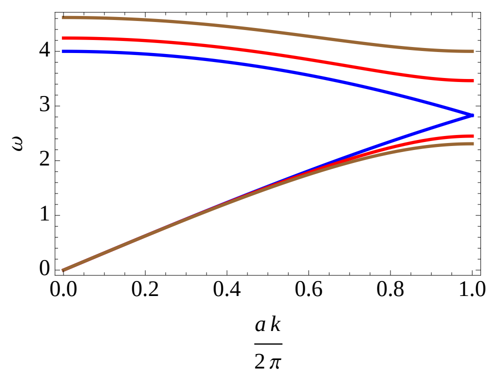
$$\omega^2 = \frac{2\alpha(1+\mu)}{m} = 2\alpha\left(\frac{1}{m} + \frac{1}{M}\right)$$

Notice that it is symmetric under the exchange $m \leftrightarrow M$, as it must be, since the speed of sound clearly cannot depend on which label we assign to which mass. It also has no k dependence. That's interesting. If $\omega > 0$ as $k \to 0$, I suppose the wave speed diverges! Let's come back to this little problem.

Taking now the negative sign Eq. (12) we get

$$\omega^2 = \frac{\alpha(1+\mu)\mu}{2m}\left(\frac{ka}{1+\mu}\right)^2 = \frac{\alpha}{2M}\frac{k^2a^2}{1+\mu}$$

$$\omega = \frac{ka}{2}\sqrt{\frac{2\alpha}{m+M}} \tag{13}$$



Therefore, the speed of sound is

$$\boxed{v = \frac{\omega}{k} = a\sqrt{\frac{\alpha}{2(m+M)}}}$$

Discussion

The figure above right shows the dispersion of a one-dimensional chain of atoms of alternating character as a function of $ka/2$ for three different values of $\mu = m/M$: $\mu = 1$ (blue), $\mu = 2$ (red), and $\mu = 3$ (brown). The curves have been scaled to have the same slope in the lower branch (called the **acoustic phonon branch**). When $m = M$, we reduce to the result derived in class for a chain of identical atoms. The upper curve corresponds to folding back the sine curve so that the peak occurs at $k = 0$, not $k = \pi/a$. This arises because the period is $a/2$ in the model of this problem, but a in the one we did before.

Something interesting happens when the masses are different, however. A gap opens up at the right edge of the graph (called the edge of the **Brillouin zone**) and the upper branch (the **optical phonon branch**) becomes disconnected from the acoustical phonon branch.

To understand what is going on here, let's return to Eq. (10) to see how the various atoms are moving. First take $k \to 0$, which sends $\cos ka/2$ to 1. Then

$$\frac{A}{B} = \rho = \frac{(\mu-1) \mp (\mu+1)}{2\mu} = \begin{cases} 1 & \text{lower sign, acoustic branch} \\ -\frac{M}{m} & \text{upper sign, optical branch} \end{cases}$$

So, on the acoustic branch the neighboring atoms move in the same direction, while on the optical branch they move in opposition. For larger values of k the atoms move in a compressional wave of wavelength $\lambda = 2\pi/k$. All the way along the acoustic branch the atoms in a unit cell (one m and the adjacent M) the atoms move in the same direction, while along the optical branch they move in opposite directions.

At the edge of the Brillouin zone, $\cos ka/2 \to 0$. You can show that

$$\rho \to \begin{cases} \frac{\mu-1}{\mu \cos ka/2} \to B = 0 & \text{lower sign, acoustic branch for } \mu > 1 \\ \frac{\cos ka/2}{\mu-1} \to A = 0 & \text{upper sign, optical branch for } \mu > 1 \end{cases}$$

That is, when $\mu > 1$ or $m > M$, the acoustic branch has the lighter atoms M at rest and the heavier atoms m vibrating in opposition to each nearest-neighbor m atom. On the optical branch, it is the heavier atoms at rest and the lighter atoms vibrating in opposition.

Problem 4 The atoms of a five-atom linear molecule of form ABABA are linked with identical springs of constant α. The mass of the A molecules is m, and the mass of the B molecules is M. We consider in this problem only the longitudinal motion of the atoms (the motion along the line joining their centers).

(a) Without calculating anything, describe the longitudinal normal modes.

(b) If q_i represents the displacement of the ith atom (counting from the left) from its equilibrium position, transform coordinates to

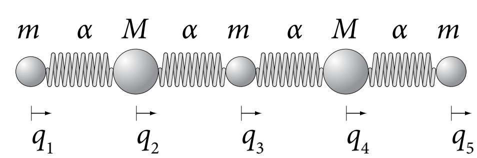
$$\eta_1 \equiv \frac{q_1 + q_5}{\sqrt{2}}, \qquad \eta_5 \equiv \frac{q_1 - q_5}{\sqrt{2}}, \qquad \eta_2 \equiv \frac{q_2 + q_4}{\sqrt{2}}, \qquad \eta_4 \equiv \frac{q_2 - q_4}{\sqrt{2}}, \qquad \eta_3 \equiv q_3$$

In terms of these variables, develop the mass matrix m_{ij} and the spring matrix A_{ij}.

(c) Solve for the eigenfrequencies (using *Mathematica* if you wish).

Solution:

The molecule is pictured at right. Let's see if we can identify the normal modes before calculating much of anything. Two modes will have the middle atom at rest. In one, atoms 1 and 2 move left while atoms 4 and 5 move symmetrically to the right. In the other, the outside atoms 1 and 5 move in while the inner atoms 2 and 4 move outward. In two other modes the middle atom vibrates. In one, atoms 2, 3, and 4 move right while the outer atoms move left; in the other, atoms 1, 3, and 5 move right while atoms 2 and 4 move left.



Let q_j be the displacement of the jth atom of mass m_j from its equilibrium position. The kinetic energy is

$$T = \frac{m}{2}(\dot{q}_1^2 + \dot{q}_3^2 + \dot{q}_5^2) + \frac{M}{2}(\dot{q}_2^2 + \dot{q}_4^2)$$

and the potential energy is

$$V = \frac{\alpha}{2}\left[(q_2 - q_1)^2 + (q_3 - q_2)^2 + (q_4 - q_3)^2 + (q_5 - q_4)^2\right]$$

Rewriting these in terms of the coordinates η_j defined in the problem statement gives

$$T = \frac{m}{2}(\dot{\eta}_1^2 + \dot{\eta}_3^2 + \dot{\eta}_5^2) + \frac{M}{2}(\dot{\eta}_2^2 + \dot{\eta}_4^2) \tag{14}$$

$$V = \frac{\alpha}{2}\left(\eta_1^2 - 2\eta_1\eta_2 + 2\eta_2^2 - 2\sqrt{2}\eta_2\eta_3 + 2\eta_3^2 + 2\eta_4^2 - 2\eta_4\eta_5 + \eta_5^2\right) \tag{15}$$

The mass and spring-constant matrices are therefore

$$m_{jk} = \begin{pmatrix} m & & & & \\ & M & & & \\ & & m & & \\ & & & M & \\ & & & & m \end{pmatrix} \quad \text{and} \quad a_{jk} = \alpha \begin{pmatrix} 1 & -1 & 0 & & \\ -1 & 2 & -\sqrt{2} & & \\ 0 & -\sqrt{2} & 2 & & \\ & & & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \tag{16}$$

The 5×5 matrices divide into an upper 3×3 and a lower 2×2 matrix. Starting with the upper 3×3,

$$\begin{vmatrix} \alpha - m\omega^2 & -\alpha & 0 \\ -\alpha & 2\alpha - M\omega^2 & -\sqrt{2}\alpha \\ 0 & -\sqrt{2}\alpha & 2\alpha - m\omega^2 \end{vmatrix} = 0$$

Divide through by α, let $X = m\omega^2/\alpha = \omega^2/\omega_0^2$, where $\omega_0^2 \equiv \alpha/m$, and let $\mu = M/m$. The determinant is then

$$\begin{vmatrix} 1 - X & -1 & 0 \\ -1 & 2 - \mu X & -\sqrt{2} \\ 0 & -\sqrt{2} & 2 - X \end{vmatrix} = -X[\mu X^2 - (3\mu + 2)X + (2\mu + 3)] = 0$$

One eigenvalue is $X = 0$, so $\omega^2 = 0$, which corresponds to a center-of-mass translation. If this is so, then all the atoms should be the same displacement. To check, substitute $X = 0$ in the (full) matrix and solve for the eigenvector. Since $X = 0$ multiplies the matrix m_{jk}, we are left with the matrix a_{jk}:

$$\begin{pmatrix} 1 & -1 & 0 & & \\ -1 & 2 & -\sqrt{2} & & \\ 0 & -\sqrt{2} & 2 & & \\ & & & 2 & -1 \\ & & & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \end{pmatrix} = 0$$

The first row implies $a = b$, which means that the amplitude of η_1 is equal to the amplitude of η_2; the second that $c = a/\sqrt{2}$. The fifth that $d = e$, and the fourth therefore that $d = e = 0$. Now, a is the coefficient of $\eta_1 = (q_1 + q_5)/\sqrt{2}$, b is the coefficient of $\eta_2 = (q_2 + q_4)/\sqrt{2}$, and $\eta_3 = q_3$. Thus, all the atoms undergo the same displacement in the mode with $\omega = 0$, as we expected.

The other two are solutions of the quadratic equation,

$$X = \frac{\omega^2}{\omega_0^2} = \frac{3\mu + 2 \pm \sqrt{9\mu^2 + 12\mu + 4 - 4\mu(2\mu + 3)}}{2\mu} = \frac{3\mu + 2 \pm \sqrt{\mu^2 + 4}}{2\mu} \tag{17}$$

Since these involve motion of η_1, η_2, and η_3, they are the modes with $\eta_3 \neq 0$, as may be confirmed by substituting these values into the matrix and solving for the eigenvectors. They are the second pair mentioned at the beginning. You may confirm that the higher-frequency mode (the plus sign in Eq. (17)) corresponds to vibration in which the atoms of mass m move left while those of mass M move right, while the lower-frequency mode has the three middle atoms moving right while the outer ones move left.

Returning to the 2×2 system, we have

$$\begin{vmatrix} 2 - \mu X & -1 \\ -1 & 1 - X \end{vmatrix} = 2 + \mu X^2 - X(2 + \mu) - 1 = 0$$

with solution

$$X = \frac{\omega^2}{\omega_0^2} = \frac{2 + \mu \pm \sqrt{\mu^2 + 4\mu + 4 - 4\mu}}{2\mu} = \frac{2 + \mu \pm \sqrt{\mu^2 + 4}}{2\mu}$$

In these modes $\eta_1 = \eta_2 = \eta_3 = 0$. Since $\eta_3 = 0$, the middle atom is not moving. The lower-frequency mode has atoms 1 and 2 moving left while atoms 4 and 5 move right; the higher-frequency mode has atoms 1 and 5 moving in while atoms 2 and 4 move out.