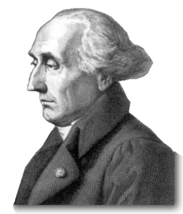
Problem Set 3 Solution

Friday, 23 September 2011

Physics 111



Problem 1 – A Bowl for cherries A particle of mass m slides without friction inside a spherical bowl of radius R. Using spherical coordinates, write down the Lagrangian. Deduce the equations of motion. Note carefully that you may neglect the size of the particle and you need not *solve* the equations of motion!

Solution: The velocity in spherical coordinates is

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}}$$

so the kinetic energy is

$$T = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)$$

and the potential energy is $U = mgz = mgR(1 - \cos\theta)$, where I have taken $\theta = 0$ at the bottom of the bowl. Therefore,

$$L = \frac{1}{2}mR^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + mgR(\cos\theta - 1)$$

The ϕ equation of motion is simple:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \qquad\Longrightarrow\qquad mR^2\sin^2\theta\,\dot{\phi} = p^{\phi} = \text{constant}$$

The equation for θ is more complicated:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \qquad\Longrightarrow\qquad mR^2\sin\theta\cos\theta\,\dot{\phi}^2 - mgR\sin\theta - \frac{d}{dt}\left(mR^2\dot{\theta}\right) = 0$$

Dividing through by $-mR^2$ we get

$$\ddot{\theta} + \frac{g}{R}\sin\theta - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\theta} + \frac{g}{R}\sin\theta - \frac{(p^{\phi})^2}{m^2R^4}\cot\theta\csc^2\theta = 0$$

Problem 2 – Loop the loop A sphere of mass m and radius a rolls without slipping inside a semicircular track of radius R that occupies a vertical plane. Let ϕ represent the angle of rotation of the sphere, and let θ measure the angle between the center of the semicircle and the center of the sphere, with respect to the vertical. That is, at $\theta = 0$, the sphere is at the bottom of the track.

(a) *Carefully* deduce the equation of constraint relating θ and ϕ. Use a large, clear diagram to clarify the argument. Hint: the equation is *not* $R\Delta\theta = -a\Delta\phi$.

(b) Noting that the rotational kinetic energy of the sphere may be expressed $\frac{1}{2}kma^2\dot{\phi}^2$, where the constant k depends on the radial dependence of the mass density (and is $2/5$ for a uniform distribution), and that the total kinetic energy is the sum of the rotational kinetic energy and the translation kinetic energy of the sphere's center of mass, write down the Lagrangian.

(c) Use the constraint equation to eliminate ϕ from L.

(d) Derive the equation of motion.

(e) Calculate the period of small oscillations about the bottom of the track by considering small displacements from equilibrium.

Solution:

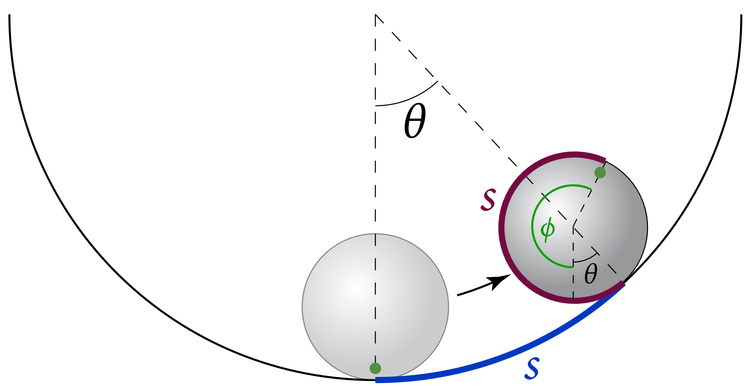


Figure 1: *As the sphere of radius a rolls along the surface of the track, the arc length along the track is equal to the arc length along the surface of the sphere.*

(a) Rolling without slipping means that the distance along the bowl, $s = R\theta$ is equal to the distance along the surface of the sphere, which is $s = a(-\phi + \theta)$, as illustrated in Fig. 1. Assuming that both θ and ϕ use the same counterclockwise direction as positive, as θ increases, ϕ decreases, which is why I have inserted the minus sign. Equating the two expressions for s, we get $-\phi = (R/a - 1)\theta$.

(b) The center of the sphere travels on a circle of radius $R - a$, so the translational kinetic energy is $\frac{1}{2}m(R - a)^2\dot{\theta}^2$. The rotational kinetic energy is $\frac{1}{2}kma^2\dot{\phi}^2$, and the gravitational potential energy is $mg(R-a)(1-\cos\theta)$. Putting these together we have

$$L = \frac{1}{2}m(R-a)^2\dot{\theta}^2 + \frac{1}{2}mka^2\dot{\phi}^2 - mg(R-a)(1-\cos\theta)$$

(c) Using the constraint equation to eliminate $\dot{\phi}$, we get

$$\begin{aligned} L &= \frac{1}{2}m(R-a)^2\dot{\theta}^2 + \frac{1}{2}mka^2\left(\frac{R-a}{a}\right)^2\dot{\theta}^2 - mg(R-a)(1-\cos\theta) \\ &= \frac{1}{2}m(1+k)(R-a)^2\dot{\theta}^2 - mg(R-a)(1-\cos\theta) \end{aligned}$$

(d)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} \qquad \Longrightarrow \qquad -mg(R-a)\sin\theta - \frac{d}{dt}\left(m(1+k)(R-a)^2\dot{\theta}\right) = 0$$

$$\ddot{\theta} + \frac{g}{(1+k)(R-a)}\sin\theta = 0$$

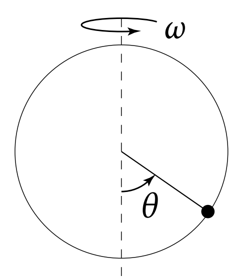
For small oscillations, we may approximate $\sin\theta \approx \theta$, giving the simple harmonic oscillator equation,

$$\ddot{\theta} + \omega^2\theta = 0$$

where $\omega = \sqrt{g/[(1+k)(R-a)]}$. Therefore, the period of small oscillations is

$$P = 2\pi\sqrt{(1+k)(R-a)/g}$$

Problem 3 – Bead on Hoop Forced to Rotate, Part I A bead of mass m is placed on a vertically oriented circular hoop of radius R which is forced to rotate with constant angular velocity ω about a vertical axis through its center, as shown.



(a) Using angle θ up from the bottom as the single generalized coordinate, write down the kinetic energy of the bead. Remember that it has motion due to the forced rotation of the hoop as well as motion due to changing θ.

(b) Find the potential energy of the bead.

(c) Find the bead's equation of motion using Lagrange's equation.

(d) Is its energy conserved? Why or why not?

(e) Find its Hamiltonian. Is H conserved? Why or why not?

(f) Is $E = H$? Why or why not?

Solution: [Note that this problem is written from the (traditional) perspective that the energy is $E = T + U$, not the perspective of Helliwell & Sahakian, which is that the energy is the Hamiltonian.] In spherical coordinates the bead's velocity is

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\sin\theta\dot{\phi}\hat{\boldsymbol{\phi}}$$

but in this case $r = R$ is a constant and $\dot{\phi} = \omega$, which is also a constant. So, the kinetic energy is

$$T = \frac{m}{2}(R^2\dot{\theta}^2 + R^2\omega^2\sin^2\theta)$$

and the potential energy is

$$U = -mgR\cos\theta$$

measured from the center of the hoop. Thus, the Lagrangian is

$$L = T - U = \frac{mR^2}{2}(\dot{\theta}^2 + \omega^2\sin^2\theta) + mgR\cos\theta$$

Lagrange's equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 = mR^2(\ddot{\theta}) - (mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta)$$

which we can simplify by dividing through by mR^2 to get the equation of motion

$$\boxed{\ddot{\theta} + \frac{g}{R}\sin\theta - \omega^2\sin\theta\cos\theta = 0}$$

(d) Energy (meaning the sum of the kinetic and potential energy) is not conserved because the hoop is forced to turn at a steady rate and can therefore work on the bead (or be worked on by the bead). More explicitly, the position of the bead is given in Cartesians by

$$\mathbf{r} = R\sin\theta\cos\omega t\,\hat{\mathbf{x}} + R\sin\theta\sin\omega t\,\hat{\mathbf{y}} - R\cos\theta\,\hat{\mathbf{z}} \tag{1}$$

which is an explicit function of the time t. Therefore, $E = T + U$ is not conserved.
(e) The Hamiltonian is given by

$$\begin{aligned} H &= \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L = \dot{\theta}\,mR^2\dot{\theta} - \frac{mR^2}{2}(\dot{\theta}^2 + \omega^2\sin^2\theta) - mgR\cos\theta \\ &= \frac{mR^2}{2}(\dot{\theta}^2 - \omega^2\sin^2\theta) - mgR\cos\theta \end{aligned}$$

It is a constant of the motion, since

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = 0$$

(f) The Hamiltonian, which is conserved, is not equal to $T + U$, which is not conserved, since this requires that the equation of transformation between generalized and Cartesian coordinates, Eq. (1), have no explicit time dependence.

Problem 4 – Bead on Hoop Forced to Rotate, Part II As in the previous problem, a bead of mass m is placed on a vertically oriented circular hoop of radius $R = 100.000$ mm which is forced to rotate with constant angular velocity $2\pi f$ about its center. Take $g = 9.800\,000\ \mathrm{m/s^2}$. Note: *Mathematica* hints are posted on the *Mathematica* page of the course web site.

(a) The bead is affixed to the hoop at $\theta = \pi/4$, and the hoop is made to rotate at $f = 2.000\,00$ Hz. If at time $t = 0$ the wax holding the bead to the hoop is flash-melted and the bead slides freely, at what time will the bead reach its greatest elevation (greatest value of θ) for the first time? What is the greatest value of θ it attains?

(b) Find the amplitude and period of the motion as a function of the angle of release, θ_0 for $f = 2.000\,000$ Hz, with θ going between $5°$ and $175°$. In particular, give values at both $5°$ and $175°$, and make a plot of the period and amplitude over the whole range. Comment briefly on anything unusual you notice in the behavior of the system.

Solution: See the end of the document.

Bead on a Hoop

Peter N. Saeta, 21 September 2009

A circular hoop of negligible mass is made to rotate about a vertical axis through its center at constant angular velocity $\omega = 2\pi f$. A small bead slides without friction along the hoop. Solve for the motion of the bead.

Revised, 23 September 2011

Graphics

Although not strictly necessary, it useful to prepare a graphical representation of the problem to see if the solution "looks right." In this case, we need to take care that the volume of the three-dimensional representation is held fixed and that the bead doesn't get cut off when it approaches the edge of the box. I'll make the bead have a radius one tenth the hoop radius, which I take to be unity. The PlotBead function takes a list of the two angles, $\{\theta, \phi\}$, as its argument and produces a graphic showing the hoop, the axis of rotation, and the bead, combined using the Show function.

```
In[1]:= PlotBead[ angles_ ] :=
 Module[{hoop, bead, ax, a = 0.1, x, y, z, θ = angles[[1]], ϕ = angles[[2]]},

  hoop =
   ParametricPlot3D[{Sin[q] Cos[ϕ], Sin[q] Sin[ϕ], -Cos[q]}, {q, 0, 2 π}, PlotStyle → Thick,
    BoxRatios → Automatic, PlotRange → {{-1.1, 1.1}, {-1.1, 1.1}, {-1.1, 1.1}}];
  ax = ParametricPlot3D[{0, 0, z}, {z, -1.1, 1.1},
    PlotRange → {{-1.1, 1.1}, {-1.1, 1.1}, {-1.1, 1.1}}];
  x = Sin[θ] Cos[ϕ];
  y = Sin[θ] Sin[ϕ];
  z = -Cos[θ];
  bead = ParametricPlot3D[{x + a Sin[q] Cos[f], y + a Sin[q] Sin[f], z + a Cos[q]},
    {q, 0, π}, {f, 0, 2 π}, Mesh → None,  BoxRatios → Automatic];
  Show[{ax, hoop, bead}, Axes → False, Boxed → True]
 ]
```

Dynamics

```
In[2]:= g = 9.8;
R = 0.1;
ssol = 1; (* this will be replaced with a solution to the DE *)
```

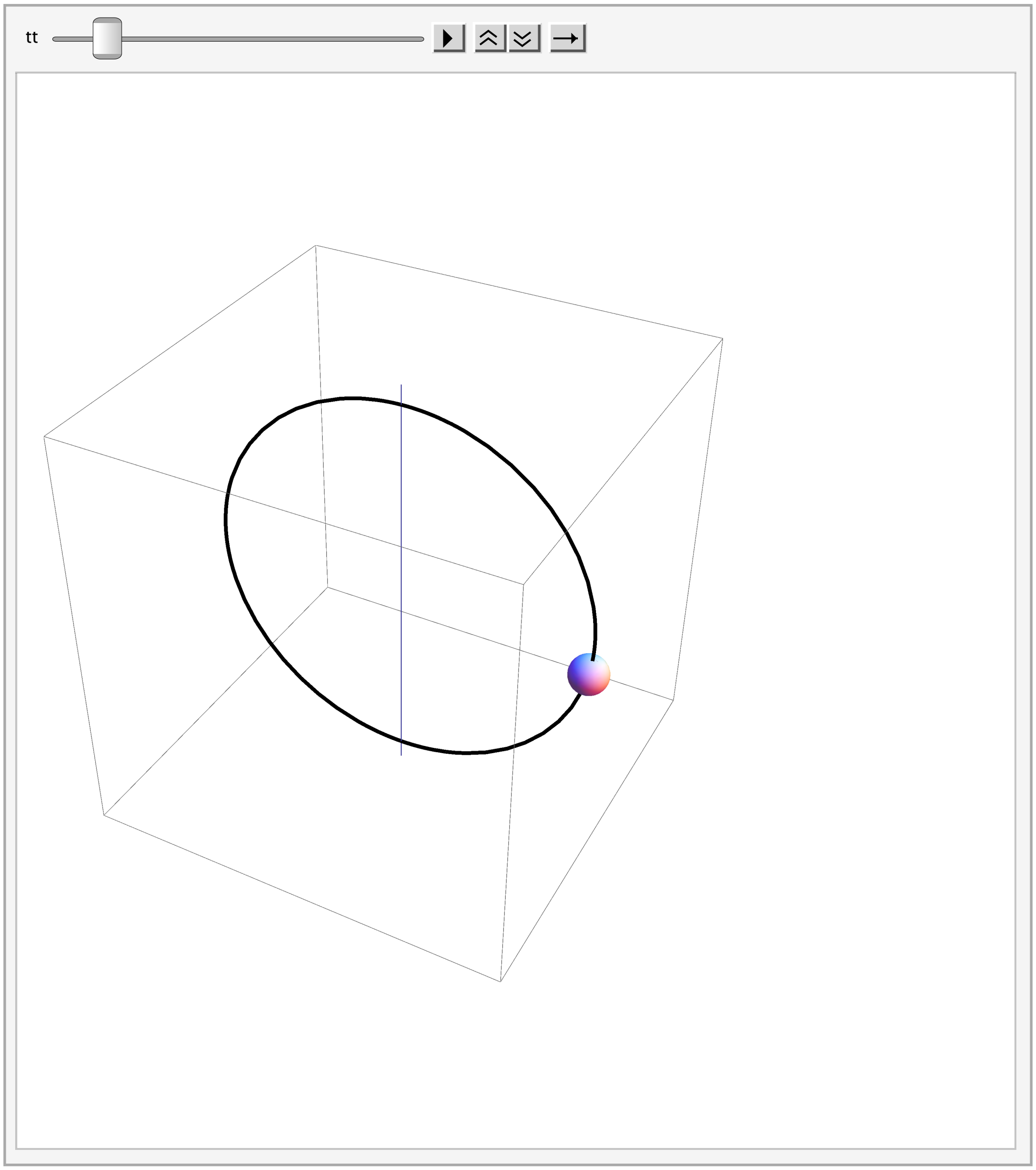
I will assume that the bead starts from rest at the beginning of the simulation time, and will simulate for a fixed time of 5 seconds (rather arbitrary, but probably sufficient for the present purpose). MakeBeadGo takes the initial angular position of the bead and the rotation frequency f of the hoop, in hertz, and computes the numerical solution to the differential equation of motion in ssol, then animates the solution.

```
In[5]:= MakeBeadGo[θ0_, f_] := Module[{ω = 2 π f},

  ssol = NDSolve[{θ''[t] + g/R Sin[θ[t]] - ω^2/2 Sin[2 θ[t]] == 0,
      θ'[0] == 0, θ[0] == θ0}, θ[t], {t, 0, 5}] // First;
  Animate[ PlotBead[ { (θ[t] /. ssol) /. t → tt, 2 π f tt}] ,
   {tt, 0, 5, 0.01}, AnimationRate → 0.25, AnimationRunning → False] ]
```

In[6]:=
```
MakeBeadGo[10 °, 2.22817]
```

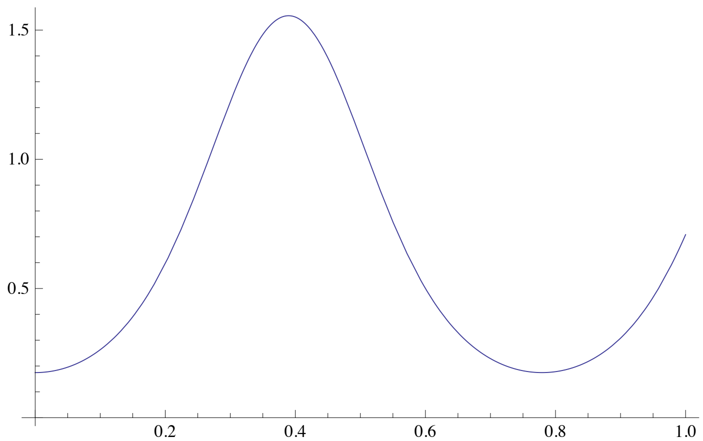
Out[6]=



To perform calculations on the numerical solution object, we apply ssol to θ[t] as follows :

In[7]:=
```
Plot[(θ[t] /. ssol), {t, 0, 1}]
```

Out[7]=



We can find the peak using the FindMaximum function, for which it is important to give a reasonable starting guess. It outputs the maximum value, and the value of the independent variable that corresponds to the maximum.

In[8]:=
```
FindMaximum[ (θ[t] /. ssol), {t, 0.2}]
```

Out[8]= $\{1.5556, \{t \to 0.389604\}\}$

To automate finding the period and amplitude, we can use a **Module**, which is a *Mathematica* procedure having local variables. The **FindAmpPeriod**[] function takes in an initial angular position and rotation rate, then integrates the equation of motion for a 3 seconds. It then looks for either the first minimum or the first maximum, depending on the slope of θ(t) at $t = 0$. Finally, it returns the amplitude (half the angular range) and the period.

In[9]:=
```
FindAmpPeriod[θ0_, f_] :=
 Module[{ω = 2 π f, s, minlist, halfPeriod, rising, extreme, amplitude, tabl, k, x},

  s = NDSolve[{θ''[t] + g/R Sin[θ[t]] - ω^2/2 Sin[2 θ[t]] == 0,
      θ'[0] == 0, θ[0] == θ0}, θ, {t, 0, 3}] // First;
  (* Figure out whether the initial condition corresponds to
   a minimum or a maximum *)
  rising = If[ (Evaluate[θ[t] /. s] /. t → 0.001) > θ0, 1, -1];

  (* To seed the FindMinimum routine, do a rough scan looking for a turning point *)
  tabl = Table[ θ[t] /. s, {t, 0.05, 3, 0.05}];

  (* Separate compound statements with a semicolon;
  a For loop takes 4 arguments in a list (separated by commas). *)
  For[x = 2, x < Length[tabl], x++,
   k = (tabl[[x]] - tabl[[x - 1]]) * (tabl[[x + 1]] - tabl[[x]]);
   If[k  < 0, Break[], ]];

  minlist = FindMaximum[ Evaluate[ rising * θ[t] /. s], {t, 0.05 x}];
  (* FindMaximum returns a list of two items: the value of the function at maximum
     and a list of rules for the independent variable(s) at the maximum *)
  halfPeriod = t /. minlist[[2]];
  extreme = minlist[[1]] rising;

  amplitude = Abs[extreme - θ0]/2;
  (* The final statement (without semicolon) in a Module is the return value *)
  {amplitude, halfPeriod 2} ]
```

In[10]:= `FindAmpPeriod[ 78 °, 2]`

Out[10]= {1.36136, 1.61424}

In[11]:= `77 ° // N`

Out[11]= 1.3439

In[12]:= `FindAmpPeriod[ π / 4, 2]`

Out[12]= {0.110991, 0.643397}

The solution

In[13]:= `FindAmpPeriod[ 5 °, 2.0 ]`

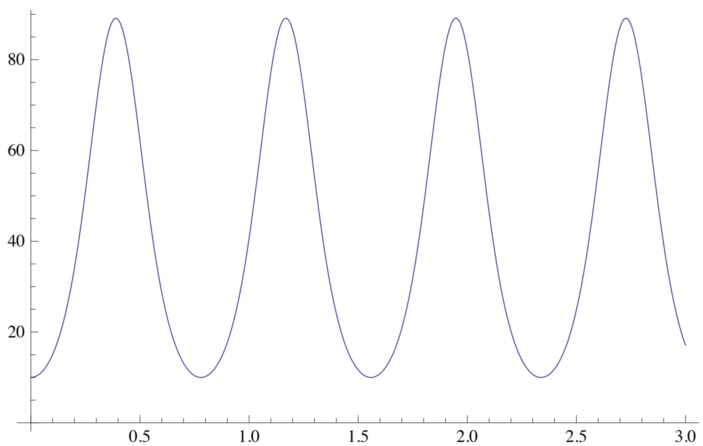
Out[13]= {0.61801, 1.10716}

In[14]:= `FindAmpPeriod[ 175 °, 2.0 ]`

Out[14]= {3.05433, 1.25157}

In[15]:= `Plot[ Evaluate[ 180 / π θ[t] /. ssol], {t, 0, 3}]`

Out[15]=



In[16]:= `{3.05433, 0.61801} 180 / π`

Out[16]= {175., 35.4094}

Looks like it's close enough. So, the amplitude at 5° is 0.61801 (35.409°) and the period is 1.10716 s. At 175° release the amplitude is 3.05433 (175°) and the period is 1.25157 s.

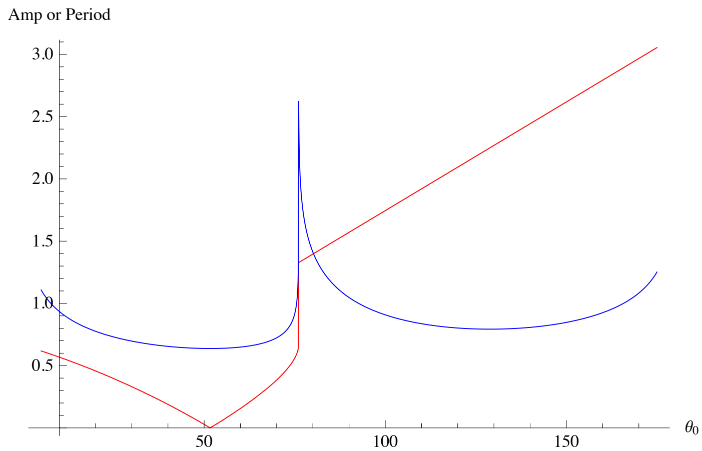
Now let' s plot the amplitude and period as functions of θ.

In[17]:= `FindAmpPeriod[1.33, 2]`

Out[17]= {1.33, 2.29099}

In[18]:=
```
Plot[ {FindAmpPeriod[ Evaluate[x °], 2.0][[1]], FindAmpPeriod[ Evaluate[x °], 2.0][[2]]},
  {x, 5, 175 } , PlotStyle → {Red, Blue}, AxesLabel → {θ0, "Amp or Period"}]
```

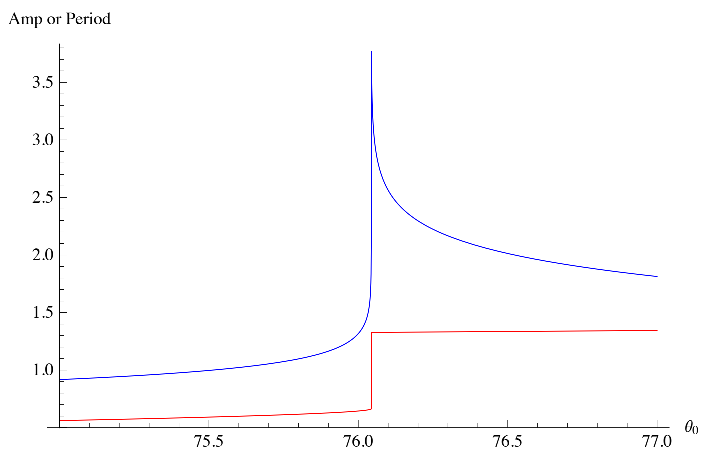
Out[18]=



Wow, that's an interesting graph. Something funky is clearly happening near 76°. Let's zoom in:

In[19]:=
```
Plot[ {FindAmpPeriod[ Evaluate[x °], 2.0][[1]], FindAmpPeriod[ Evaluate[x °], 2.0][[2]]},
  {x, 75, 77 } , PlotStyle → {Red, Blue}, AxesLabel → {θ0, "Amp or Period"}]
```

Out[19]=



In[20]:= ss[θ0_] := θ[t] /. NDSolve$\left[\left\{\theta''[t] + \frac{g}{R}\,\mathrm{Sin}[\theta[t]] - \frac{(4\,\pi)^2}{2}\,\mathrm{Sin}[2\,\theta[t]] == 0,\ \theta'[0] == 0,\ \theta[0] == \theta0\right\}, \theta[t], \{t, 0, 5\}\right]$

In[21]:=
```
Plot[ Evaluate[ {ss[ 76 °], ss[76.05 °], ss[ 76.1 °]}], {t, 0, 5}]
```

Out[21]=