Chapter 6

Gravitation and Central-force motion

In this chapter we describe motion caused by central forces, especially the orbits of planets, moons, and artificial satellites due to central **gravitational** forces. Historically, this is the most important testing ground of Newtonian mechanics. In fact, it is not clear how the science of mechanics would have developed if the Earth had been covered with permanent clouds, obscuring the Moon and planets from view. And Newton's laws of motion with central gravitational forces are still very much in use today, such as in designing spacecraft trajectories to other planets. Our treatment here of motion in central gravitational forces is followed in the next chapter with a look at motion due to **electromagnetic** forces. But before we end the current chapter, we take a brief look at Einstein's **general theory of relativity**, which built upon his special theory and the principle of equivalence to lead to a fully relativistic theory of gravity in which gravity is not a force at all, but an effect of spacetime curvature.

6.1 Central forces

A *central* force on a particle is directed toward or away from a fixed point in three dimensions and is spherically symmetric about that point. In spherical coordinates (r, θ, φ) the corresponding potential energy is also spherically symmetric, with $U = U(r)$ alone.

For example, the Sun, of mass m_1 (the source), exerts an attractive central force

$$\mathbf{F} = -G\frac{m_1\, m_2}{r^2}\hat{\mathbf{r}} \tag{6.1}$$

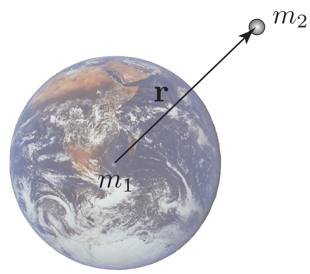

Figure 6.1: Newtonian gravity pulling a probe mass m_2 towards a source mass m_1.

on a planet of mass m_2 (the probe), where r is the distance between their centers and $\hat{\mathbf{r}}$ is a unit vector pointing away from the Sun (see Figure 6.1). The corresponding gravitational potential energy is

$$U(r) = -\int F(r)\, dr = -G\frac{m_1\, m_2}{r}\ . \tag{6.2}$$

Similarly, the spring-like central force from a fixed point (the source) to the mass it is attached to (the probe)

$$\mathbf{F} = -k\mathbf{r} = -k\, r\ \hat{\mathbf{r}} \tag{6.3}$$

and has a three-dimensional spring potential energy

$$U(r) = -\int F(r)\, dr == \frac{1}{2}k\, r^2\ . \tag{6.4}$$

And the Coulomb force

$$\mathbf{F} = \frac{q_1\, q_2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{6.5}$$

on a charge q_2 (the probe) due to a central charge q_1 (the source) has a Coulomb potential energy

$$U(r) = -\int F(r)\, dr = \frac{1}{4\pi\epsilon_0}\frac{q_1\, q_2}{r}\ . \tag{6.6}$$

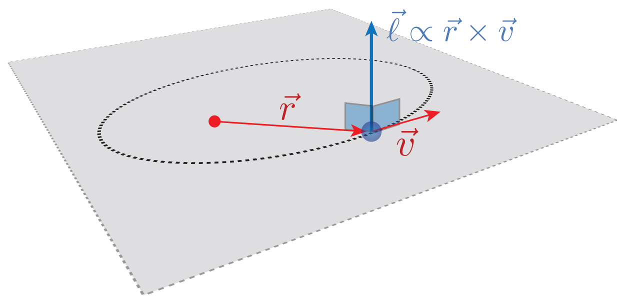

Figure 6.2: Angular momentum conservation and the planar nature of central force orbits.

In all these cases, the force is along the direction of the line joining the centers of the source point and the probe object.

The environment of a particle subject to a central force is invariant under rotations about any axis through the fixed point at the origin, so the angular momentum $\boldsymbol{\ell}$ of the particle is conserved, as we saw in Chapters 4 and 5. Conservation of $\boldsymbol{\ell}$ also follows from the fact that the torque $\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F} = 0$ due to a central force, if the fixed point is chosen as the origin of coordinates. The particle therefore moves in a *plane*,[1]because its position vector $\mathbf{r}$ is perpendicular to the fixed direction of $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$ (see Figure 6.2). Hence, central force problems are necessarily two-dimensional.

All this discussion assumes that the source of the central force is fixed in position: the Sun, or the pivot of the spring, or the source charge q_1 are all not moving and lie at the origin of our coordinate system. What if the source object is also in motion? If it is accelerating, as is typically the case due to the reaction force exerted on it by the probe, the source then defines a non-inertial frame and cannot be used as a reference point of our coordinates along with Newton's second

[1]The plane in which a particle moves can also be defined by two vectors: (i) the radius vector to the particle from the force center, and (ii) the initial velocity vector of the particle. Given these two vectors, as long as the central force remains the *only* force, the particle cannot move out of the plane defined by these two vectors. (We are assuming that the two vectors are noncolinear; if $\mathbf{r}$ and $\mathbf{v}_0$ are parallel or antiparallel the motion is obviously only one-dimensional, along a radial straight line.)

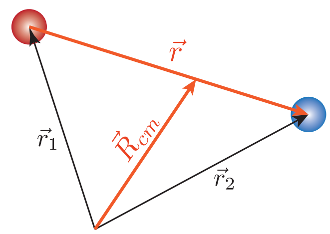

Figure 6.3: The classical two-body problem in physics.

law. Let us then proceed to tackling the more general situation, the so-called two-body problem involving two dynamical objects, both moving around, pulling on each other through a force that lies along the line that joins their centers.

6.2 The two-body problem

In this section we show that with the right choice of coordinates, the two-body problem is equivalent to a one-body central-force problem. If we can solve the one-body central-force problem, we can solve the two-body problem.

In the two-body problem there is a kinetic energy for each body and a mutual potential energy that depends only upon the distance between them. There are altogether six coordinates, three for the first body, $\mathbf{r}_1 = (x_1, y_1, z_1)$, and three for the second, $\mathbf{r}_2 = (x_2, y_2, z_2)$, where all coordinates are measured from a fixed point in some inertial frame (see Figure 6.3). The alternative set of six coordinates used for the two-body problem are, first of all, three **center of mass** coordinates

$$\mathbf{R}_{cm} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \tag{6.7}$$

already defined in Section 1.3: The CM vector extends from a fixed point in some inertial frame to the center of mass of the bodies. There are also three **relative coordinates**

$$\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1, \tag{6.8}$$

where the relative coordinate vector points from the first body to the second, and its length is the distance between them.

We can solve for $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{R}_{cm}$ and $\mathbf{r}$:

$$\mathbf{r}_1 = \mathbf{R}_{cm} - \frac{m_2}{M}\mathbf{r} \quad \text{and} \quad \mathbf{r}_2 = \mathbf{R}_{cm} + \frac{m_1}{M}\mathbf{r}, \tag{6.9}$$

where $M = m_1 + m_2$ is the total mass of the system. The total kinetic energy of the two bodies, using the original coordinates for each, is[2]

$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2, \tag{6.10}$$

which can be reexpressed in terms of the new generalized velocities $\dot{\mathbf{R}}_{\mathbf{cm}}$ and $\dot{\mathbf{r}}$. The result is (See Problem 6.10)

$$T = \frac{1}{2}M\dot{\mathbf{R}}_{cm}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 \tag{6.11}$$

where

$$\mu = \frac{m_1 m_2}{M} \tag{6.12}$$

is called the **reduced mass** of the two-body system (note that μ is less than either m_1 or m_2.) The mutual potential energy is $U(r)$, a function of the distance r between the two bodies. Therefore the Lagrangian of the system can be written

$$L = T - U = \frac{1}{2}M\dot{\mathbf{R}}_{cm}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r) \tag{6.13}$$

in terms of $\mathbf{R}_{cm}$, $\mathbf{r}$, and their time derivatives. One of the advantages of the new coordinates is that the coordinates $\mathbf{R}_{\mathbf{CM}} = (X_{cm}, Y_{cm}, Z_{cm})$ are cyclic, so the corresponding total momentum of the system $\mathbf{P} = M\dot{\mathbf{R}}_{\mathbf{CM}}$ is conserved. That is, the center of mass of the two-body system drifts through space with constant momentum and constant velocity.

The remaining portion of the Lagrangian is

$$L = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - U(r) = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) - U(r), \tag{6.14}$$

which has the same form as that for a single particle orbiting around a force center, written in polar coordinates. We already know that this problem is entirely

[2]Note that we adopt the linear algebra notation for a square of a vector $\mathbf{V}$: $\mathbf{V}^2 \equiv \mathbf{V}\cdot\mathbf{V} = |\mathbf{V}|^2 = \mathrm{V}^2$.

two-dimensional since the angular momentum vector is conserved. We can then choose our spherical coordinates so that the plane of the dynamics corresponding to $\theta = \varphi/2$. This allows us to write a simpler Lagrangian with two degrees of freedom only

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - U(r), \tag{6.15}$$

We then immediately identify two constants of the motion:

(i) L is not an explicit function of time, so the Hamiltonian H is conserved, which in this case is also the sum of kinetic and potential energies:

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) = \text{constant}. \tag{6.16}$$

(ii) The angle θ is cyclic, so the corresponding generalized momentum p^θ, which we recognize as the angular momentum of the particle, is also conserved:

$$p^\theta \equiv \ell = \mu\, r^2\dot{\theta} = r\,(\mu r\dot{\theta}) = \text{constant}. \tag{6.17}$$

This is the magnitude of the conserved angular momentum vector $\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}$, written in our coordinate system, with $\mathbf{p} = \mu\mathbf{v}$.

With only two degrees of freedom left over r and θ, the two conservation laws of energy and angular momentum together form a complete set of first integrals of motion for a particle moving in response to a central force or in a two-body problem. We will proceed in the next section by solving the problem at hand explicitly in two different ways.

Before we do this however, let us note an interesting attribute of our setup. Our original two-body problem collapsed into a two-dimensional one-body problem described through a position vector $\mathbf{r}$ pointing from the source m_1 to the probe m_2. This position vector traces out the *relative* motion of the probe about the source. Yet the source may be moving around and accelerating. Although it may appear that one is incorrectly formulating physics from the perspective of a potentially non-inertial frame — that of the source — this is not so. The elegance of the two-body central force problem arises in part from the fact that the information about the non-inertial aspect of the source's perspective is neatly tucked into one parameter, μ: we are describing the relative motion of m_2 with respect to m_1 by tracing out the trajectory of a fictitious particle of mass $\mu = m_1 m_2/(m_1 + m_2)$ about m_1. Our starting point Lagrangian of the two-body problem was written from the perspective of a third entity, an inertial observer. Yet, after a sequence of coordinate transformations and simplifications, we have found the problem is mathematically equivalent to describing the dynamics of an object of mass μ about the source mass m_1.

Note also that if we are in a regime where the source mass is much heavier than the probe, $m_1 \gg m_2$, we then have approximately $\mu \simeq m_2$. In such a scenario, the source mass m_1 is too heavy to be affected much by m_2's pull. Hence, m_1 is staying put and is indeed an inertial frame, and m_2 is going around it. In this regime, we recover the naive interpretation that one is tracing out the relative motion of a probe mass m_2 from the perspective of an inertial observer sitting with m_1.

6.3 The effective potential energy

We start by analyzing the dynamics qualitatively, and in some generality. To do this, we use energy diagrams to get a feel of the possible types of trajectories the probe would trace out. The two conservation equations are

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \quad \text{and} \quad \ell = \mu r^2\dot{\theta}, \tag{6.18}$$

which we can use to eliminate between them the angle θ instead of the time t. Instead of concentrating on the shape of orbits, we are now concerned with the time it takes to move from one radius to another. From the angular momentum conservation equation, we have $\dot{\theta} = \ell/mr^2$, so then energy conservation gives

$$\frac{1}{2}\mu\dot{r}^2 + U_{\text{eff}}(r) = E \tag{6.19}$$

where the "effective potential energy" is

$$U_{\text{eff}}(r) \equiv \frac{\ell^2}{2\mu r^2} + U(r). \tag{6.20}$$

Angular momentum conservation has allowed us to convert the *rotational* portion of the kinetic energy $(1/2)\mu r^2\dot{\theta}^2$ into a term $\ell^2/2\mu r^2$ that depends on position alone, so it behaves just like a potential energy. Then the sum of this term and the "real" potential energy $U(r)$ (which is related to the central force $F(r)$ by $F(r) = -dU(r)/dr$) together form the effective potential energy. The extra term is often called the "centrifugal potential"

$$U_{\text{cent}}(r) \equiv \frac{\ell^2}{2\mu r^2} \tag{6.21}$$

because its corresponding "force" $F_{\text{cent}} = -dU_{\text{cent}}/dr = +\ell^3/\mu r^3$ tends to push the orbiting particle away from the force center at the origin. By eliminating θ between the two conservation laws, they combine to form an equation that *looks* like a one-dimensional energy conservation law in the variable r. So as long as we add in the centrifugal potential energy, we can use all our experience with one-dimensional conservation-of-energy equations to understand the motion. In general, we can tell that if our $U_{\text{eff}}(r)$ has a minimum

$$U'_{\text{eff}}\big|_{r=R} = -\frac{\ell^2}{\mu r^3} + U'(r)\Big|_{r=R} = 0 \ , \tag{6.22}$$

the system admits circular orbits at $r = R$. Such an orbit would be stable if $U''_{\text{eff}} > 0$, unstable if $U''_{\text{eff}} < 0$, and critically stable if $U''_{\text{eff}} = 0$. This translates to conditions of the form

$$-3\frac{\ell^2}{\mu r^4} + U''(r)\Big|_{r=R} \begin{cases} > 0 & \text{Stable} \\ < 0 & \text{Unstable} \\ = 0 & \text{Critically stable} \end{cases} \tag{6.23}$$

We can also determine whether the system admits bounded non-circular orbits — where $r_{\text{min}} < r < r_{\text{max}}$ — or unbounded orbits — where r can extend all the way to infinity. Let us look at a couple of examples to see how the effective energy diagram method can be very useful.

6.3.1 Radial motion for the central-spring problem

The effective potential energy of a particle in a central-spring potential is

$$U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} + \frac{1}{2}k\,r^2, \tag{6.24}$$

which is illustrated (for $\ell \neq 0$) in Figure 6.4. At large radii the attractive spring force $F_{\text{spring}} = -dU(r)/dr = -k\,r$ dominates, but at small radii the centrifugal potential takes over, and the associated "centrifugal force", given by $F_{\text{cent}} = -dU_{\text{cent}}/dr = \ell^2/\mu r^3$ is positive, and therefore outward, an inverse-cubed strongly repulsive force. We can already tell that this system admits **bounded** orbits: for every orbit, there is a minimum and maximum values of r for the dynamics. In this case, we will see that these bounded orbits are also closed. That is, after a 2π's worth of evolution in θ, the probe traces back the same trajectory. To find the explicit shape of these trajectories — which will turn out to be ellipses — we will need to integrate our differential equations. We will come back to this in Section 6.5. For now, we can already answer interesting questions such as the time of travel for the probe to move between two radii. Solving equation (6.19) (with $U_{\text{eff}} = \ell^2/2\mu r^2 + k\,r^2$) for $\dot{r}^2$ and taking the square root gives

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - k\,r^2 - \frac{\ell^2}{2\mu r^2}\right)}. \tag{6.25}$$

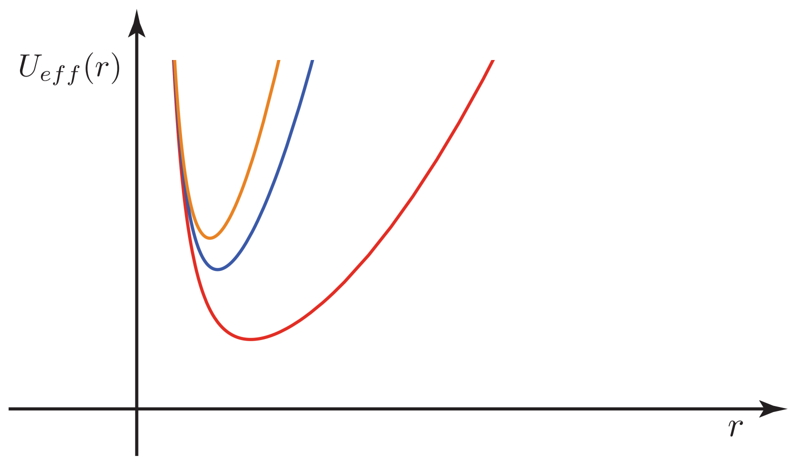

Figure 6.4: The effective potential for the central-spring potential.

Separating variables and integrating,

$$t(r) = \pm \, \frac{\mu}{2} \int_{r_0}^{r} \frac{r \; dr}{\sqrt{Er^2 - k\, r^4 - \ell^2/2\mu}}, \tag{6.26}$$

where we choose $t = 0$ at some particular radius r_0. We have reduced the problem to quadrature. In fact, the integral can be carried out analytically[3].

6.3.2 Radial motion in central gravity

The effective potential energy of a particle in a central gravitational field is

$$U_{\text{eff}}(r) = \frac{\ell^2}{2mr^2} - \frac{GMm}{r}, \tag{6.27}$$

which is illustrated (for $\ell \neq 0$) in Figure 6.5. At large radii the inward gravitational force $F_{\text{grav}} = -dU(r)/dr = -GMm/r^2$ dominates, but at small radii the centrifugal potential takes over, and the associated "centrifugal force", given by $F_{\text{cent}} = -dU_{\text{cent}}/dr = \ell^2/mr^3$ is positive, and therefore outward, an inverse-cubed strongly repulsive force that pushes the planet away from the origin if it gets too close.

Two very different types of orbit are possible in this potential. There are **bound** orbits with energy $E < 0$, and **unbound** orbits, with energy $E \geq 0$. Bound

[3] See Problem 6-15?

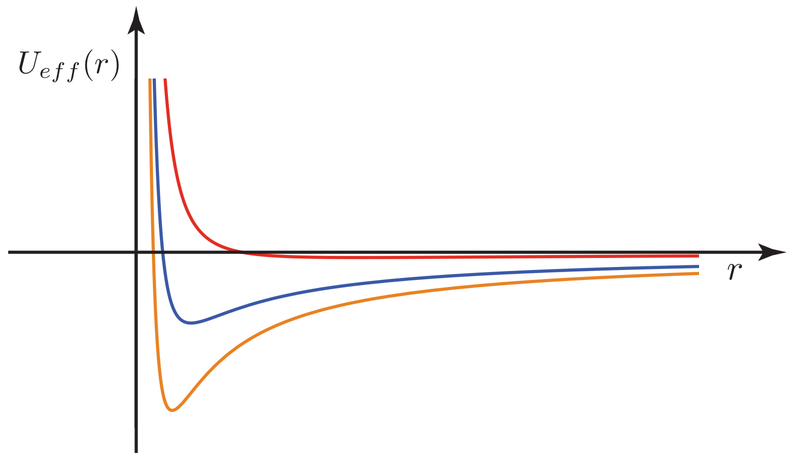

Figure 6.5: The effective gravitational potential.

orbits do not escape to infinity. They include circular orbits with an energy E_{min} corresponding to the energy at the bottom of the potential well, where only one radius is possible, and there are orbits with $E > E_{\text{min}}$ (but with E still negative), where the planet travels back and forth between inner and outer turning points while it is also rotating about the center. The minimum radius is called the **periapse** for orbits around an arbitrary object, and specifically the **perihelion, perigee,** and **periastron** for orbits around the Sun, the Earth, and a star. The maximum radius, corresponding to the right-hand turning point, is called the **apoapse** in general, or specifically the **aphelion, apogee,** and **apastron**.

Unbound orbits are those where there is no outer turning point: these orbits extend out infinitely far. There are orbits with $E = 0$ that are just barely unbound: in this case the kinetic energy goes to zero in the limit as the orbiting particle travels infinitely far from the origin. And there are orbits with $E > 0$ where the particle still has nonzero kinetic energy as it escapes to infinity. In fact, we will see that orbits with energies $E = E_{\text{min}}$ are circles, those with $E_{\text{min}} < E < 0$ are ellipses, those with $E = 0$ are parabolas, and those with $E > 0$ are hyperbolas.

Now we can tackle the effective one-dimensional energy equation in (r, t) to try to obtain a second integral of motion. Our goal is to find $r(t)$ or $t(r)$, so we will know how far a planet, comet, or spacecraft moves radially in a given length of time, or long it takes any one of them to travel between two given radii in its orbit.

Solving equation (6.19) (with $U_{\text{eff}} = \ell^2/2\mu r^2 - G\,m_1 m_2/r$) for $\dot{r}^2$ and taking the square root gives

$$\frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E + \frac{Gm_1m_2}{r} - \frac{\ell^2}{2\mu r^2}\right)}. \tag{6.28}$$

Separating variables and integrating,

$$t(r) = \pm\,\frac{\mu}{2}\int_{r_0}^{r}\frac{r\;dr}{\sqrt{Er^2 + G\,m_1m_2r - \ell^2/2\mu}}, \tag{6.29}$$

where we choose $t = 0$ at some particular radius r_0. We have reduced the problem to quadrature. In fact, the integral can be carried out analytically, so we can calculate how long it takes a planet or spacecraft to travel from one radius to another in its orbit (See Problem 6-15?).

6.4 Bertrand's Theorem

In the previous section, we saw central potentials that admit bounded and unbounded orbits. Bounded orbit are of particular interest since they can potentially *close*. That is, after a certain finite number of revolution, the probe starts tracing out its established trajectory — thus closing its orbit. A beautiful and powerful result of mechanics is theorem due J. Bertrand which states the following:

The only central force potentials $U(r)$ for which all *bounded orbits are closed are the following:*

1. *The gravitational potential $U(r) \propto 1/r$.*
2. *The central-spring potential $U(r) \propto r^2$.*

The theorem asserts that, of all possible functional forms for a potential $U(r)$, only two kinds lead to the interesting situation where all bounded orbits close! And these two potentials are familiar ones. The theorem is not very difficult to prove. We leave it to the Problems section at the end of the chapter.

6.5 The *shape* of central-force orbits

We will first eliminate the time t from the equations, leaving only r and θ. This will allow us to find orbital *shapes*. That is, we will find a single differential equation involving r and θ alone, which will give us a way to find the shape $r(\theta)$, the radius of the orbit as a function of the angle, or $\theta(r)$, the angle as a function of the radius.

Beginning with the first integrals

$$E = \frac{1}{2}m\dot{r}^2 + \frac{\ell^2}{2mr^2} + U(r) \qquad \text{and} \qquad \ell = mr^2\dot{\theta}, \tag{6.30}$$

we have two equations in the three variables, r, θ, and t. When finding the shape $r(\theta)$ we are unconcerned with the time it takes to move from place to place, so we eliminate t between the two equations. Solving for dr/dt in the energy equation and dividing by $d\theta/dt$ in the angular momentum equation,

$$\frac{dr}{d\theta} = \frac{dr/dt}{d\theta/dt} = \pm\sqrt{\frac{2m}{\ell^2}}r^2\sqrt{E - \ell^2/2mr^2 - U(r)}, \tag{6.31}$$

neatly eliminating t. Separating variables and integrating,

$$\theta = \int d\theta = \pm\frac{\ell}{\sqrt{2m}}\int^r \frac{dr/r^2}{\sqrt{E - \ell^2/2mr^2 - U(r)}} \tag{6.32}$$

reducing the shape problem to quadrature. Further progress in finding $\theta(r)$ requires a choice of $U(r)$.

6.5.1 Central spring-force orbits

A spring force $\mathbf{F} = -k\mathbf{r}$ pulls on a particle of mass m toward the origin at $r = 0$. The force is central, so the particle moves in a plane with a potential energy $U = (1/2)kr^2$. What is the shape of its orbit? From equation (6.32),

$$\theta(r) = \pm\frac{\ell}{\sqrt{2m}}\int^r \frac{dr/r^2}{\sqrt{E - \ell^2/2mr^2 - (1/2)kr^2}}. \tag{6.33}$$

Multiplying top and bottom of the integrand by r and substituting $z = r^2$ gives

$$\theta(z) = \pm\frac{\ell}{2\sqrt{2m}}\int^z \frac{dz/z}{\sqrt{-\ell^2/2m + Ez - (k/2)z^2}}. \tag{6.34}$$

On the web or in integral tables we find that

$$\int^z \frac{dz/z}{\sqrt{a + bz + cz^2}} = \frac{1}{\sqrt{-a}}\sin^{-1}\left(\frac{bz + 2a}{z\sqrt{b^2 - 4ac}}\right) \tag{6.35}$$

where a, b, and c are constants, with $a < 0$. In our case $a = -\ell^2/2m, b = E$, and $c = -k/2$, so

$$\begin{aligned}\theta - \theta_0 &= \pm\frac{\ell}{2\sqrt{2m}}\frac{1}{\sqrt{\ell^2/2m}}\sin^{-1}\left(\frac{bz+2a}{z\sqrt{b^2-4ac}}\right)\\ &= \pm\frac{1}{2}\sin^{-1}\left(\frac{Er^2-\ell^2/m}{r^2\sqrt{E^2-k\ell^2/m}}\right)\end{aligned} \tag{6.36}$$

where θ_0 is a constant of integration. Multiplying by ± 2, taking the sine of each side, and solving for r^2 gives the orbital shape equation

$$r^2(\theta) = \frac{\ell^2/m}{E \mp (\sqrt{E^2 - k\ell^2/m})\sin 2(\theta-\theta_0)}. \tag{6.37}$$

Note that the orbit is closed (since $r^2(\theta + 2\pi) = r^2(\theta)$), and that it has a long axis (corresponding to an angle θ where the denominator is small because the second term subtracts from the first term) and a short axis (corresponding to an angle where the denominator is large, because the second term adds to the first term.) In fact, the shape $r(\theta)$ is that of an **ellipse** with $r = 0$ at the *center* of the ellipse.[4]

The orbit is illustrated in Figure 6.6 for the case $\theta_0 = 0$ and with a minus sign in the denominator. The effect of changing the sign or using a nonzero θ_0 is simply to rotate the entire figure about its center, while keeping the "major" axis and the "minor" axis perpendicular to one another.

6.5.2 The shape of gravitational orbits

By far the most important orbital shapes are for central gravitational forces. This is the problem that Johannes Kepler wrestled with in his self-described "War on Mars." Equipped with the observational data on the positions of Mars from Tycho Brahe, he tried one shape after another to see what would fit, beginning with a circle (which didn't work), various ovals (which didn't work), and finally an ellipse (which did.) Now we can derive the shape by two different methods, by solving the integral of equation (6.32), and (surprisingly enough!) by *differentiating*

[4]A common way to express an ellipse in polar coordinates with $r = 0$ at the center is to orient the major axis horizontally and the minor axis vertically, which can be carried out by selecting the plus sign in the denominator and choosing $\theta_0 = \pi/4$ (See Problem 6.11). In this case the result can be written $r^2 = a^2b^2/(b^2\cos^2\theta + a^2\sin^2\theta)$ where a is the semimajor axis (half the major axis) and b is the semiminor axis. In Cartesian coordinates ($x = r\cos\theta, y = r\sin\theta$) this form is equivalent to the common ellipse equation $x^2/a^2 + y^2/b^2 = 1$.

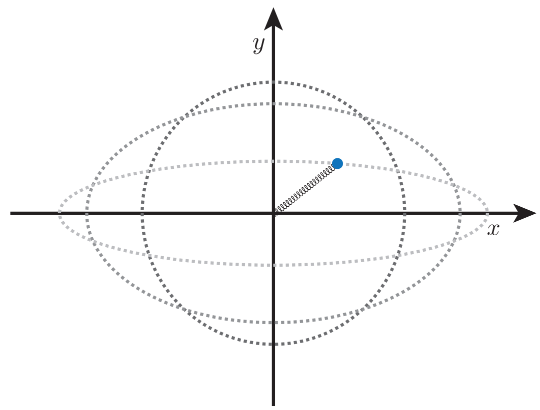

Figure 6.6: Elliptical orbits due to a central spring force $\mathbf{F} = -k\mathbf{r}$.

equation (6.31).

By direct integration

For a central gravitational force the potential energy $U(r) = -GMm/r$, so the integral for $\theta(r)$ becomes

$$\theta = \int d\theta = \pm \frac{\ell}{\sqrt{2m}} \int \frac{dr/r}{\sqrt{Er^2 + GMmr - \ell^2/2m}} \tag{6.38}$$

which by coincidence is the same integral we encountered in Example 6-1 (using there the variable $z = r^2$ instead),

$$\int \frac{dr/r}{\sqrt{a + br + cr^2}} = \frac{1}{\sqrt{-a}} \sin^{-1}\left(\frac{br + 2a}{r\sqrt{b^2 - 4ac}}\right), \tag{6.39}$$

where now $a = -\ell^2/2m, b = GMm$, and $c = E$. Therefore

$$\theta - \theta_0 = \pm \sin^{-1}\left(\frac{GMm^2 - \ell^2}{\epsilon\; GMm^2 r}\right), \tag{6.40}$$

where θ_0 is a constant of integration and we have defined the **eccentricity**

$$\epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}}. \tag{6.41}$$

We will soon see the geometrical meaning of ϵ. Taking the sine of $\theta-\theta_0$ and solving for r gives

$$r = \frac{\ell^2/GMm^2}{1 \pm \epsilon \sin(\theta - \theta_0)}. \tag{6.42}$$

By convention we choose the plus sign in the denominator together with $\theta_0 = \pi/2$, which in effect locates $\theta = 0$ at the point of closest approach to the center, called the **periapse** of the ellipse. This choice changes the sine to a cosine, so

$$r = \frac{\ell^2/GMm^2}{1 + \epsilon \cos\theta}. \tag{6.43}$$

This equation gives the allowed shapes of orbits in a central gravitational field. Before identifying these shapes, we will derive the same result by a very different method that is often especially useful.

By differentiation

Returning to equation (6.31) with $U(r) = -GMm/r$,

$$\frac{dr}{d\theta} = \pm\sqrt{\frac{2m}{\ell}} r^2 \sqrt{E - \frac{\ell^2}{2mr^2} + \frac{GMm}{r}}, \tag{6.44}$$

we will now *differentiate* rather than *integrate* it. The result turns out to be greatly simplified if we first introduce the *inverse radius* $u = 1/r$ as the coordinate. Then

$$\frac{dr}{d\theta} = \frac{d(1/u)}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}. \tag{6.45}$$

Squaring this gives

$$\left(\frac{du}{d\theta}\right)^2 \equiv (u')^2 = \frac{2m}{\ell^2}\left(E - \frac{\ell^2 u^2}{2m} + (GMm)u\right). \tag{6.46}$$

Differentiating both sides with respect to θ,

$$2u'u'' = -2uu' + \frac{2GMm^2}{\ell^2}u'. \tag{6.47}$$

Then dividing out the common factor u', since (except for a circular orbit) it is generally nonzero, we find

$$u'' + u = \frac{GMm^2}{\ell^2}. \tag{6.48}$$

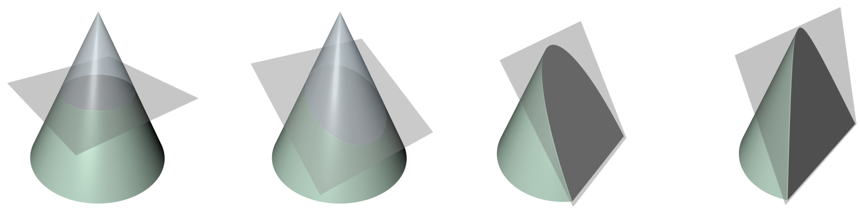

Figure 6.7: Conic sections: circles, ellipses, parabolas, and hyperbolas.

The most general solution of this *linear* second-order differential equation is the sum of the general solution of the homogeneous equation $u'' + u = 0$ and any particular solution of the full (inhomogeneous) equation. The most general solution of $u''+u = 0$ can be written $u_H = A\cos(\theta-\delta)$, where A and δ are the two required arbitrary constants. A particular solution of the full equation is the constant $u_P = GMm^2/\ell^2$, so the general solution of the full equation is

$$u = A\cos(\theta - \delta) + GMm^2/\ell^2. \tag{6.49}$$

The shape of the orbit is therefore

$$r = 1/u = \frac{\ell^2/GMm^2}{1 + e\cos\theta}, \tag{6.50}$$

where $\epsilon \equiv A\ell^2/GMm^2$, and we have set $\delta = 0$ so that again r is a minimum at $\theta = 0$. Equation (6.50) is the same as equation (6.43), the result we found previously by direct integration. Even though it has merely reproduced a result we already knew, the "trick" of substituting the inverse radius works for inverse-square forces, and will be a useful springboard later when we perturb elliptical orbits.

The shapes $r(\theta)$ given by equation (6.50) are known as "conic sections", since they correspond to the possible intersections of a plane with a cone, as illustrated in Figure 6.7. There are only four possible shapes: (i) circles, (ii) ellipses, (iii) parabolas, and (iv) hyperbolas. The shape equation can be rewritten in the form

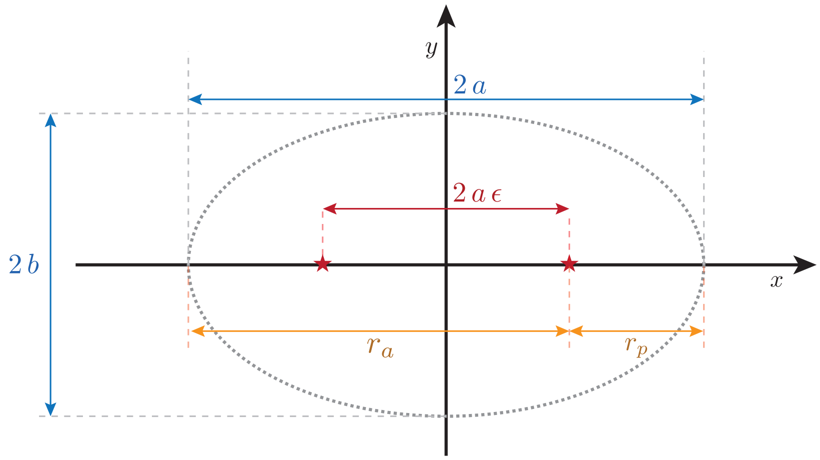

Figure 6.8: An elliptical gravitational orbit, showing the foci, the semimajor axis a, semiminor axis b, the eccentricity ϵ, and the periapse and apoapse.

$$r = \frac{r_p(1+\epsilon)}{1+\epsilon\cos\theta} \tag{6.51}$$

where r_p is the point of closest approach of the orbit to a fixed point called the **focus**.

1. For **circles**, the eccentricity $\epsilon = 0$, so the radius $r = r_p$, a constant independent of angle θ. The focus of the orbit is at the center of the circle.

2. For **ellipses**, the eccentricity obeys $0 < \epsilon < 1$. Note from the shape equation that in this case, as with a circle, the denominator cannot go to zero, so the radius remains finite for all angles. There are two foci in this case, and r_p is the closest approach to the focus at the right in Figure 6.8, where the angle $\theta = 0$. *Note that the force center at $r = 0$ is located at one of the foci of the ellipse for the gravitational force, unlike the ellipse for a central spring force of Example 6-1, where the force center was at the center of the ellipse.*

   The long axis of the ellipse is called the major axis, and half of this distance is the semimajor axis, denoted by the symbol a. The semiminor axis, half of the shorter axis, is denoted by b. One can derive several properties of ellipses from equation (6.43) in this case.

   (a) The **periapse** and **apoapse** of the ellipse (the closest and farthest points of the orbit from the right-hand focus) are given, in terms of

a and ϵ, by $r_p = a(1-\epsilon)$ and $r_a = a(1+\epsilon)$, respectively. Therefore equation (6.50) can be written in the alternative form

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}. \tag{6.52}$$

(b) The sum of the distances d_1 and d_2 from the two foci to a point on the ellipse is the same for all points on the ellipse.[5]

(c) The distance between the two foci is $2a\epsilon$, so the eccentricity of an ellipse is the ratio of this interfocal distance to the length of the major axis.

(d) The semiminor and semimajor axes are related by $b = a\sqrt{1-\epsilon^2}$.

(e) The area of the ellipse is $A = \pi ab$.

3. For **parabolas**, the eccentricity $\epsilon = 1$, so $r \to \infty$ as $\theta \to \pm\pi$, and the shape is as shown in Figure 6.9. One can show that every point on a parabola is equidistant from a focus and a line called the **directrix**, also shown on the figure.

4. For **hyperbolas**, the eccentricity $\epsilon > 1$, so $r \to \infty$ as $\cos\theta \to -1/\epsilon$. This corresponds to two angles, one between $\pi/2$ and π, and one between $-\pi/2$ and $-\pi$, as shown in Figure 6.9.

EXAMPLE 6-1: Orbital geometry and orbital physics

Now we can relate the geometrical parameters of a gravitational orbit, the eccentricity ϵ and semimajor axis a, to the physical parameters, the energy E and angular momentum ℓ. The relationship follows from the two formulas for $r(\theta)$ we have written, namely

$$r = \frac{\ell^2/GMm^2}{1+\epsilon\cos\theta} \qquad \text{and} \qquad r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta} \tag{6.53}$$

[5] Therefore the well-known property of an ellipse, that it can be drawn on a sheet of paper by sticking two straight pins into the paper some distance D apart, and dropping a loop of string over the pins, where the loop has a circumference greater than $2D$. Then sticking a pencil point into the loop as well, and keeping the loop taut, moving the pencil point around on the paper, the resulting drawn figure will be an ellipse.

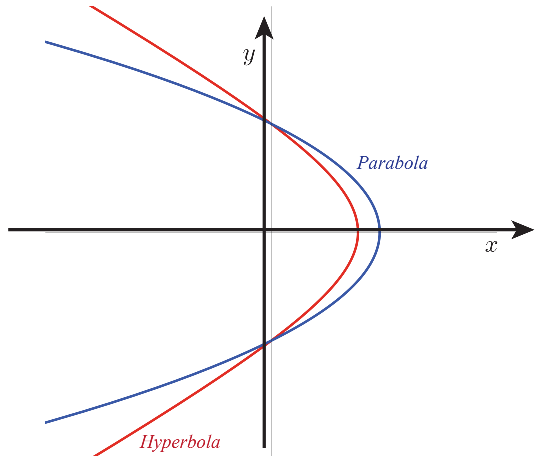

Figure 6.9: Parabolic and hyperbolic orbits

where

$$\epsilon = \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}}. \tag{6.54}$$

We first consider circles and ellipses, and then parabolas and hyperbolas.

For ellipses or circles the equations match up if $a(1-\epsilon^2) = \ell^2/GMm^2$, so the semimajor axis of an ellipse (or the radius of the circle) is related to the physical parameters by

$$a = \frac{\ell^2/GMm^2}{1-\epsilon^2} = \frac{\ell^2/GMm^2}{1-(1+2E\ell^2/G^2M^2m^3)} = -\frac{GMm}{2E}, \tag{6.55}$$

depending upon E but not ℓ. In summary, for ellipses and circles the geometrical parameters a, ϵ are related to the physical parameters by

$$a = -\frac{GMm}{2E} \qquad \text{and} \qquad \epsilon = \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}}. \tag{6.56}$$

These can be inverted to give the physical parameters in terms of the geometrical parameters:

$$E = -\frac{GMm}{2a} \qquad \text{and} \qquad \ell = \sqrt{GMm^2a(1-\epsilon^2)}. \tag{6.57}$$

For parabolas and hyperbolas, the equations match if we let $r_p(1+\epsilon) = \ell^2/GMm^2$, where $\epsilon = 1$ for parabolas and $\epsilon > 1$ for hyperbolas. So the geometric parameters (r_p, ϵ) for these orbits are given in terms of the physical parameters E and ℓ by

$$r_p = \frac{\ell^2}{(1+\epsilon)GMm^2} \qquad \epsilon = \sqrt{1 + \frac{2E\ell^2}{G^2M^2m^3}}, \tag{6.58}$$

and inversely

$$E = \frac{GMm(\epsilon - 1)}{2r_p} \qquad \ell = \sqrt{GMm^2 r_p(1+\epsilon)} \tag{6.59}$$

in terms of (r_p, e). Note that for parabolas, the eccentricity $\epsilon = 1$, so the energy $E = 0$.

Finally, to summarize orbits in a central inverse-square gravitational field, note that *there are four, and only four, types of orbits possible*, as illustrated in Figure 6.10. There are **circles** ($\epsilon = 0$), **ellipses** ($0 < \epsilon < 1$), **parabolas** ($\epsilon = 1$), and **hyperbolas** ($\epsilon > 1$), with the gravitating object at one focus. Ellipses and circles are closed, bound orbits with negative total energy. Hyperbolas and parabolas are open, unbound orbits, which extend to infinity. Parabolic orbits have zero total energy, and hyperbolic orbits have a positive total energy. Circles (with $\epsilon = 0$) and parabolas (with $E = 0$) are so unique among the set of all solutions that mathematically one can say that they form "sets of measure zero", and physically one can say that they never occur in nature. The orbits of planets, asteroids, and some comets are elliptical; other comets may move in hyperbolic orbits. There are no other orbit shapes for a central gravitational field.[6] There are no "decaying" or "spiralling" purely gravitational orbits, for example.

6.6 Orbital dynamics

In the early 1600's, a young German scientist by the name of Johannes Kepler (1571-1630) — the first theoretical physicist, in the modern sense of this term — formulate three laws of celestial mechanics that revolutionized the scientific world. Pondering over data collected by Tycho Brahe (1546-1601) — the first

[6]There are also straight-line paths falling directly toward or away from the central object, but these are really limiting cases of ellipses, parabolas, and hyperbolas. They correspond to motion with angular momentum $\ell = 0$, so the eccentricity $\epsilon = 1$. If the particle's energy is negative this is the limiting case of an ellipse as $\epsilon \to 1$, if the energy is positive it is the limiting case of a hyperbola as $\epsilon \to 1$, and if the energy is zero it is a parabola with both $\epsilon = 1$ and $p^\theta = 0$.

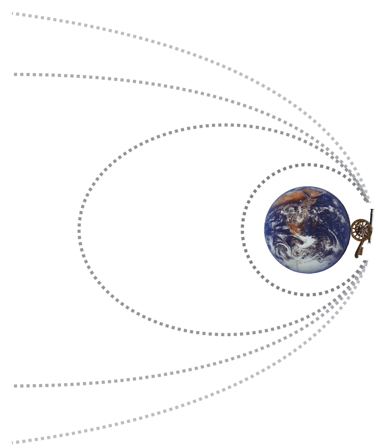

Figure 6.10: The four types of gravitational orbits

experimental physicists — Kepler identified three rules that govern the dynamics of planets and comets in the heavens:

1. *Planets move in elliptical orbits, with the Sun at one focus.*
2. *Planetary orbits sweep out equal areas in equal times.*
3. *The periods squared of planetary orbits are proportional to their semimajor axes cubed.*

It took about a century to finally understand the physical origins of these three laws through the work of Isaac Newton. Armed with new powerful tools in mechanics, we indeed confirm the first law of Kepler. To understand the second and third, we will need to do a bit more work.

6.6.1 Kepler's second law

There is an interesting consequence of angular momentum conservation for arbitrary central forces. Take a very thin slice of pie extending from the origin to the orbit of the particle, as shown in Figure 6.11. To a good approximation, becoming exact in the limit as the slice gets infinitely thin, the area of the slice is that of a triangle, $\Delta A = (1/2)$ (base $\times$ height) $= (1/2)r(r\Delta\theta) = (1/2)r^2\Delta\theta$. If the particle moves through angle $\Delta\theta$ in time Δt, then $\Delta A/\Delta t = (1/2)r^2\Delta\theta/\Delta t$, so in the limit $\Delta t \to 0$,

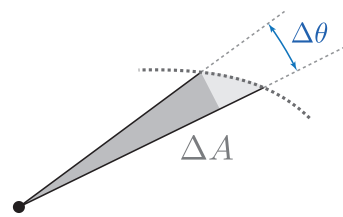

Figure 6.11: The area of a thin pie slice

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{\mu r^2\dot{\theta}}{2\mu} = \frac{\ell}{2\mu} = \text{constant}, \tag{6.60}$$

since ℓ is constant. Therefore this *areal velocity*, the rate at which area is swept out by the orbit, remains constant as the particle moves. This in turn implies that the orbit *sweeps out equal areas in equal times*. Between t_1 and t_2, for example,

$$A = \int_{t_1}^{t_2} \left(\frac{dA}{dt}\right) dt = \int_{t_1}^{t_2} \left(\frac{\ell}{2\mu}\right) dt = \left(\frac{\ell}{2\mu}\right)(t_2 - t_1), \tag{6.61}$$

which is the same as the area swept out between times t_3 and t_4 if $t_4 - t_3 = t_2 - t_1$.

And hence we have derived Kepler's second law. Kepler himself of course did not know *why* the law is true; the concepts of angular momentum and central forces had not yet been invented. In the orbit of the Earth around the Sun, for example, the areas swept out in any 31-day month, say January, July, or October, must all be the same. To make the areas equal, in January, when the Earth is closest to the Sun, the pie slice must be fatter than in July, when the Earth is farthest from the Sun. Note that the tangential velocity $v_{\rm tan}$ must be greater in January to cover the greater distance in the same length of time, which is consistent with conservation of the angular momentum $\ell = mr^2\dot{\theta} = mrv_{\rm tan}$.

Although it was first discovered for orbiting planets, the equal-areas-in-equal-times law is also valid for particles moving in *any* central force, including asteroids, comets, and spacecraft around the Sun; the Moon and artificial satellites around the Earth; and particles subject to a central attractive spring force or a hypothetical central exponential repulsive force.

6.6.2 Kepler's third law

Now we can find the period of elliptical orbits in central gravitational fields. How long does it take planets to orbit the Sun? And how long does it take the Moon, and orbiting spacecraft or other Earth satellite to orbit the Earth?

From equation (6.61) in Section 6.1, the area traced out in time $t_2 - t_1$ is $A = (\ell/2m)(t_2 - t_1)$. The period of the orbit, which is the time to travel around the entire ellipse, is therefore

$$T = (2m/\ell)A_{\text{total}} = (2m/\ell)\pi ab = \frac{2m\pi a^2\sqrt{1-e^2}}{\sqrt{GMm^2a(1-e^2)}} \tag{6.62}$$

since the area of the ellipse is $A_{\text{total}} = \pi ab = a^2\sqrt{1-e^2}$, and $\ell = \sqrt{GMm^2a(1-e^2)}$. The formula for the period simplifies to give

$$T = \frac{2\pi}{\sqrt{GM}}a^{3/2}. \tag{6.63}$$

It is interesting that the period depends upon the semimajor axis of the orbit, but *not* upon the eccentricity. Two orbits with the same semimajor axis have the same period, even though their eccentricities are different.

And we thus arrived at Kepler's third law: The periods squared of planetary orbits are proportional to their semimajor axes cubed. That is, $T^2 \propto a^3$.

EXAMPLE 6-2: Halley's Comet

This most famous comet is named after the English astronomer, mathematician, and physicist Sir Edmund Halley (1656-1742), who calculated its orbit. The comet has been known as far back as 240 BC and probably longer, and was later thought to be an omen when it appeared earlier in the year of the Norman conquest at the Battle of Hastings in the year 1066. Mark Twain was born in 1835 in one of its appearances, and predicted (correctly) that he would die in its next appearance in 1910. It last passed through the Earth's orbit in 1986 and will again in 2061.

From the comet's current period[7] of $T = 75.3$ yrs and observed perihelion distance $r_p = 0.586$ AU (which lies between the orbits of Mercury and Venus), we can calculate the orbit's (a) semimajor axis a, (b) aphelion distance r_a, and

[7]The period has varied considerably over the centuries, because the comet's orbit is easily influenced by the gravitational pulls of the planets, especially Jupiter and Saturn.

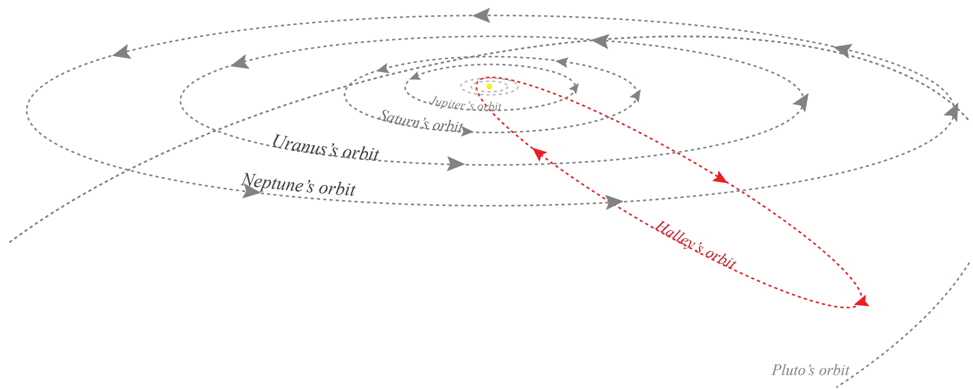

Figure 6.12: The orbit of Halley's comet

(c) eccentricity ϵ. (Note that 1 AU is the length of the semimajor axis of Earth's orbit, 1 AU = 1.5 $\times 10^{11}$ m.)

(a) From Kepler's third law, which applies to comets in bound orbits as well as to all planets and asteroids, we can compare the period of Halley's comet to the period of Earth's orbit: $T_H/T_E = (a_H/a_E)^{3/2}$, so the semimajor axis has length

$$a_H = a_E(T_H/T_E)^{2/3} = 1 \text{ AU}(75.3\text{yrs}/1\text{yr})^{2/3} = 17.8 \text{ AU}. \tag{6.64}$$

(b) The major axis is therefore 2×17.8 AU = 35.6 AU, so the aphelion distance is at $r_a = 35.8$ AU $- r_p = 35.6$ AU - 0.6 AU = 35.0 AU from the Sun. Halley's Comet retreats farther from the Sun than the orbit of Neptune.

(c) The perihelion distance is $r_p = a(1 - e)$, so the eccentricity of the orbit is

$$\epsilon = 1 - r_p/a = 0.967. \tag{6.65}$$

The orbit is highly eccentric, as you would expect, since the aphelion is thirty-six times as far from the Sun as the perihelion.

The orbit of Halley's comet is inclined at about 18^o to the ecliptic, i.e., at about 18^o to the plane of Earth's orbit, as shown in Figure 6.12. It is also retrograde; the comet orbits the Sun in the opposite direction from that of the planets, orbiting clockwise rather than counterclockwise looking down upon the solar system from above the Sun's north pole.

6.6.3 Minimum-energy transfer orbits

What is the best way to send a spacecraft to another planet? Depending upon what one means by "best", many routes are possible. But the trajectory requiring the *least fuel* (assuming the spacecraft does not take advantage of "gravitational assists" from other planets along the way, which we will discuss later), is a so-called *minimum-energy transfer orbit* or "Hohmann" transfer orbit, which takes full advantage of Earth's motion to help the spacecraft get off to a good start.

Typically the spacecraft is first lifted into low-Earth orbit (LEO), where it circles the Earth a few hundred kilometers above the surface. Then at just the right time the spacecraft is given a velocity boost "delta v" that sends it away from the Earth and into an orbit around the Sun that reaches all the way to its destination. Once the spacecraft coasts far enough from Earth that the Sun's gravity dominates, the craft obeys all the central-force equations we have derived so far, including Kepler's laws: In particular, it coasts toward its destination in an elliptical orbit with the Sun at one focus.

Suppose that in LEO the rocket engine boosts the spacecraft so that it ultimately attains a velocity v_∞ away from the Earth. Then if the destination is Mars or one of the outer planets, it is clearly most efficient if the spacecraft is aimed so that this velocity v_∞ is in the *same* direction as Earth's velocity around the Sun, because then the velocity of the spacecraft in the Sun's frame will have its largest possible magnitude, $v_E + v_\infty$. The subsequent transfer orbit towards an outer planet is shown in Figure 6.13. The elliptical path is tangent to the Earth's orbit at launch and tangent to the destination planet's orbit at arrival, just barely making it out to where we want it.

First we will find out how *long* it will take the spacecraft to reach its destination, which is easily found using Kepler's third law. Note that the major axis of the craft's orbit is $2a_C = r_E + r_P$, assuming the Earth E and destination planet P move in nearly circular orbits. The semimajor axis of the transfer orbit is therefore

$$a_C = \frac{r_E + r_P}{2}. \tag{6.66}$$

From the third law, the period T_C of the craft's elliptical orbit obeys $(T_C/T_E)^2 = (a_C/r_E)^3$, in terms of the period T_E and radius r_E of Earth's orbit. The spacecraft travels through only half of this orbit on its way from Earth to the planet, however, so the travel time is

$$T = T_C/2 = \frac{1}{2}\left(\frac{r_E + r_P}{2r_E}\right)^{3/2} T_E \tag{6.67}$$

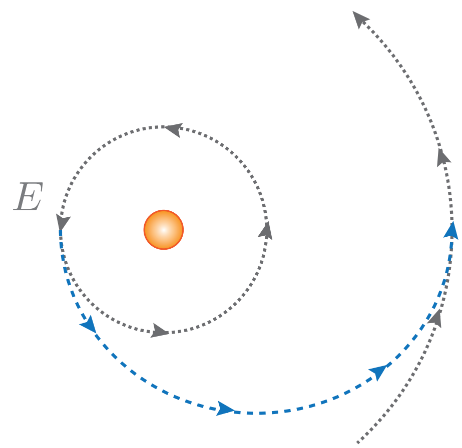

Figure 6.13: A minimum-energy transfer orbit to an outer planet.

which we can easily evaluate, since every quantity on the right is known. Now we can outline the steps required for the spacecraft to reach Mars or an outer planet.

(1) We first place the spacecraft in a parking orbit of radius r_0 around the Earth. Ideally, the orbit will be in the same plane as that of the Earth around the Sun, and the rotation direction will also agree with the direction of Earth's orbit. Using $\mathbf{F} = m\mathbf{a}$ in the radial direction, the speed v_0 of the spacecraft obeys

$$\frac{GM_E m}{r_0^2} = ma = \frac{mv_0^2}{r_0}, \tag{6.68}$$

so $v_0 = \sqrt{GM_E/r_0}$.

(2) Then at just the right moment, a rocket provides a boost Δv in the same direction as v_0, so the spacecraft now has an instantaneous velocity $v_0 + \Delta v$, allowing it to escape from the Earth in the most efficient way. This will take the spacecraft from LEO into a *hyperbolic* orbit relative to the Earth, since we want the craft to escape from the Earth with energy to spare, as shown in Figure 6.14. Then as the spacecraft travels far away, its potential energy $-GM_E m/r$ due to Earth's gravity approaches zero, so its speed approaches v_∞, where, by energy conservation,

$$\frac{1}{2}mv_\infty^2 = \frac{1}{2}m(v_0 + \Delta v)^2 - \frac{GM_E m}{r_0}. \tag{6.69}$$

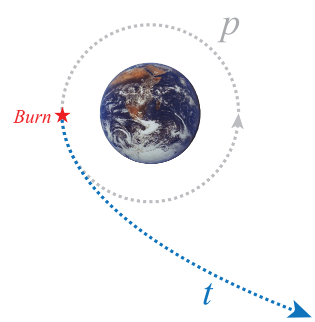

Figure 6.14: Insertion from a parking orbit into the transfer orbit.

Solving for v_∞,

$$v_\infty = \sqrt{(v_0 + \Delta v)^2 - 2GM_E/r_0} = \sqrt{(v_0 + \Delta v)^2 - 2v_0^2}. \quad (6.70)$$

This is the speed of the spacecraft relative to the Earth by the time it has essentially escaped Earth's gravity, but before it has moved very far from Earth's orbit around the Sun.

(3) Now if we have provided the boost Δv at just the right time, when the spacecraft is moving in just the right direction, by the time the spacecraft has escaped from the Earth its velocity v_∞ relative to the Earth will be in the *same* direction as Earth's velocity v_E around the Sun, so the spacecraft's velocity in the Sun's frame of reference will be as large as it can be for given v_∞,

$$v = v_\infty + v_E = \sqrt{(v_0 + \Delta v)^2 - 2v_0^2} + v_E. \quad (6.71)$$

The Earth has now been left far behind, so the spacecraft's trajectory from here on is determined by the Sun's gravity alone. We have given it the largest speed v we can in the Sun's frame for given boost Δv, to get it off to a good start.

(4) The velocity v just calculated will be the speed of the spacecraft at the perihelion point of some elliptical Hohmann transfer orbit. What speed must this be for the transfer orbit to have the desired semimajor axis a? We can find out by equating the total energy (kinetic plus potential) of the spacecraft in orbit around

the Sun with the specific energy it has in an elliptical orbit with the appropriate semimajor axis a. That is,

$$E = T + U = \frac{1}{2}mv^2 - \frac{GMm}{r} = -\frac{GMm}{2a}, \tag{6.72}$$

where m is the mass of the spacecraft, M is the mass of the Sun, r is the initial distance of the spacecraft from the Sun (which is the radius of Earth's orbit), and a is the semimajor axis of the transfer orbit. Solving for v^2, we find

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right), \tag{6.73}$$

which is known historically as the *vis-viva* equation.[8] The quantities on the right are known, so we can calculate v, which is the Sun-frame velocity the spacecraft must achieve.

EXAMPLE 6-3: A trip to Mars

We will use this scenario to plan a trip to Mars by Hohmann transfer orbit. First, we can use Kepler's third law to find how long it will take for the spacecraft to arrive. The major axis of the spacecraft's orbit is $2a_C = r_E + r_M$, assuming the Earth and Mars move in nearly circular orbits. The semimajor axis is therefore

$$a_C = \frac{r_E + r_M}{2} = \frac{1.50 + 2.28}{2} \times 10^8 \text{ km} = 1.89 \times 10^8 \text{ km}. \tag{6.74}$$

The spacecraft travels through only half of this complete elliptical orbit on its way out to to Mars, so the travel time is[9]

$$T = T_C/2 = \frac{1}{2}\left(\frac{1.89}{1.50}\right)^{3/2} (1 \text{ year}) = 258 \text{ days}. \tag{6.75}$$

[8] *Vis-viva* means "living force", a term used by the German mathematician Gottfried Wilhelm Leibniz in a now-obsolete theory. The term survives only in orbital mechanics.

[9] In his science fiction novel "Stranger in a Strange Land", Robert Heinlein looks back on the first human journeys to Mars: "an interplanetary trip ... had to be made in free-fall orbits — from Terra to Mars, 258 Terran days, the same for return, plus 455 days waiting at Mars while the planets crawled back into positions for the return orbit."

Now we will find the boost required in low-Earth orbit to insert the spacecraft into the transfer orbit. We will first find the speed required of the spacecraft in the Sun's frame just as it enters the Hohmann ellipse. From the *vis-viva* equation,

$$v = \sqrt{GM\left(\frac{2}{r} - \frac{1}{a}\right)} = 32.7 \text{ km/s}, \tag{6.76}$$

using $G = 1.67 \times 10^{-11}$ m^3/kg s^2, $M = 1.99 \times 10^{30}$ kg, $r = 1.50 \times 10^8$ km, and $a = 1.89 \times 10^8$ km. Compare this with the speed of the Earth in its orbit around the Sun,[10] $v_E = \sqrt{GM/r} = 29.7$ km/s.

Now suppose the spacecraft starts in a circular parking orbit around the Earth, with radius $r_0 = 7000$ km corresponding to an altitude above the surface of about 600 km. The speed of the spacecraft in this orbit is $v_0 = \sqrt{GM_E/r_0} = 7.5$ km/s. We then require that v_∞, the speed of the spacecraft relative to the Earth after it has escaped from the Earth, is $v_\infty = v - v_E = 32.7$ km/s - 29.7 km/s = 3.0 km/s. Solving finally for Δv in equation (6.71), we find that the required boost for this trip is

$$\begin{aligned}\Delta v &= \sqrt{v_\infty^2 + 2v_0^2} - v_0 \\ &= \sqrt{(3.0 \text{ km/s})^2 + 2(7.4 \text{ km/s})^2} - 7.5 \text{ km/s} = 3.5 \text{ km/s}.\end{aligned} \tag{6.77}$$

This boost of 3.5 km/s is modest compared with the boost needed to raise the spacecraft from Earth's surface up to the parking orbit in the first place. Then once the spacecraft reaches Mars, the rocket engine must provide an additional boost to insert the spacecraft into a circular orbit around Mars, or even to allow it to strike Mars's atmosphere at a relatively gentle speed, because the spacecraft, when it reaches the orbit of Mars in the Hohmann transfer orbit, will be moving considerably more slowly than Mars itself in the frame of the Sun. Note that the Hohmann transfer orbit will definitely take the spacecraft out to Mars orbit, but there are only limited launch windows; we have to time the trip just right so that Mars will actually be at that point in its orbit when the spacecraft arrives.

EXAMPLE 6-4: Gravitational assists

[10] Earth's speed around the Sun actually varies from 29.28 km/s at aphelion to 30.27 km/s at perihelion. It is not surprising that the spacecraft's speed of 32.7 km/s exceeds v_E; otherwise it could not escape outwards toward Mars against the Sun's gravity.

There is no more useful and seemingly magical application of the Galilean velocity transformation of Chapter 1 than *gravitational assists.* Gravitational assists have been used to send spacecraft to destinations they could not otherwise reach because of limited rocket fuel capabilities, including voyages to outer planets like Uranus and Neptune using gravitational assists from Jupiter and Saturn, and complicated successive visits to the Galilean satellite of Jupiter, gravitationally bouncing from one to another.

Suppose we want to send a heavy spacecraft to Saturn, but it has only enough room for fuel to make it to Jupiter. If the timing is just right and the planets are also aligned just right, it is possible to aim for Jupiter, causing the spacecraft to fly just *behind* Jupiter as it swings by that planet. Jupiter can pull on the spacecraft, turning its orbit to give it an increased velocity in the Sun's frame of reference, sufficient to propel it out to Saturn.

The key here is "in the Sun's frame of reference", because in Jupiter's rest frame the trajectory of the spacecraft can be turned, but there can be no net change in speed before and after the encounter. When the spacecraft is still far enough from Jupiter that Jupiter's gravitational potential energy can be neglected, the spacecraft has some initial speed v_0 in Jupiter's rest frame. As it approaches Jupiter, the spacecraft speeds up, the trajectory is bent, and the spacecraft then slows down again as it leaves Jupiter, once again approaching speed v_0. In Jupiter's own rest frame, Jupiter cannot cause a net increase in the spacecraft's speed.

However, because of the deflection of the spacecraft, its speed *can* increase in the *Sun's* rest frame, and this increased speed therefore gives the spacecraft a larger total energy in the Sun's frame, perhaps enough to project it much farther out into the solar system.

Consider a special case to see how this works. Figure 6.15(a) shows a picture of a spacecraft's trajectory in the rest frame of Jupiter. The spacecraft is in a hyperbolic orbit about Jupiter, entering from below the picture and being turned by (we will suppose) a 90^o angle by Jupiter. It enters with speed v_0 from below, and exits at the same speed v_0 toward the left. It has gained no energy in Jupiter's frame. Figure 6.15(b) shows the same trajectory drawn in the Sun's frame of reference. In the Sun's frame, Jupiter is moving toward the left with speed v_J, so the spacecraft's speed when it enters from beneath Jupiter (i.e., as it travels away from the Sun, which is much farther down in the figure) can be found by vector addition: It is $v_{\text{initial}} = \sqrt{v_0^2 + v_J^2}$, since v_0 and v_J are perpendicular to one another. However, the spacecraft's speed when it *leaves* Jupiter is $v_{\text{final}} = v_0 + v_J$, since in this case the vectors are parallel to one another. Obviously $v_{\text{final}} > v_{\text{initial}}$; the spacecraft has been sped up in the Sun's frame of reference, so that it has more energy than before, and may be able to reach Saturn as a result.

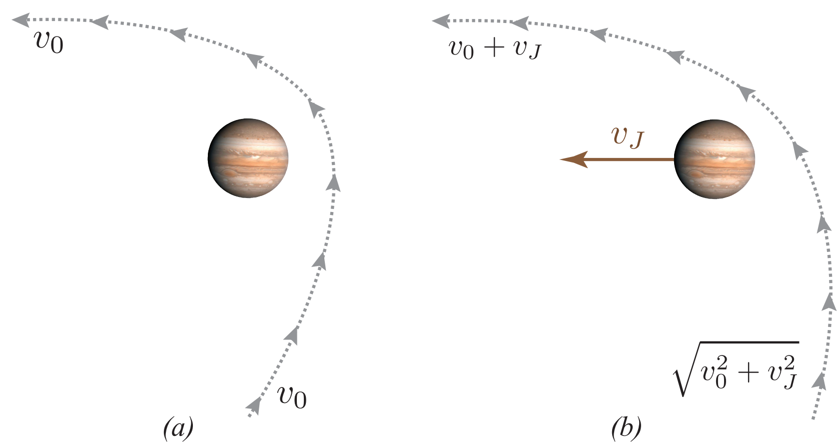

Figure 6.15: A spacecraft flies by Jupiter, in the reference frames of (a) Jupiter (b) the Sun

Clearly the trajectory must be tuned very carefully to get the right angle of flyby so that the spacecraft will be thrown in the right direction and with the right speed to reach its final destination.

6.7 Relativistic gravitation

It was already clear to Einstein in 1905 that his special theory of relativity is consistent with Maxwell's electromagnetism, but *not* with Newton's gravitation. The fundamental problem is that while the equation of motion $\mathbf{F} = -(GMm/r^2)\hat{\mathbf{r}} = m\mathbf{a}$ is invariant under the **Galilean** transformation of Chapter 1, it is *not* invariant under the relativistic **Lorentz** transformation of Chapter 2.[11] So if the equation were true in one inertial frame of reference it would not be true in another using the correct transformation, and so could not be a fundamental law of physics according to the principle of relativity.

Therefore Einstein set out to find a *relativistic* theory of gravitation. His decade-long effort finally culminated in his greatest single achievement, the **general theory of relativity**. There were several clues that gradually led him on. One was the apparently trivial fact that according to $\mathbf{F} = -GMm/r^2\hat{\mathbf{r}} = m\mathbf{a}$, the mass m cancels out on both sides: all masses m have the same acceleration in

[11] As just one indication of this, the distance r between M and m is not invariant under a Lorentz boost. See examples in Chapter 5 for more details on how to explore the transformations and symmetries of Lagrangians.

a given gravitational field according to Newton's theory and experiment as well. This is not as trivial as it seems, however, because the two m's have very different meanings. The m in GMm/r^2 is called the **gravitational mass**; it is the property of a particle that causes it to be attracted by another particle. The m in $m\mathbf{a}$ is called the **inertial mass**; it is the property of a particle that makes it sluggish, resistant to acceleration. The fact that these two kinds of mass appear to be the same is consistent with Newton's theory, but not explained by it. Einstein wanted a *natural* explanation. As discussed at the end of Chapter 3, this sort of thinking generated in Einstein his "happiest thought", the **principle of equivalence**. The equivalence of gravitational and inertial masses is an immediate consequence of the equivalence of (i) a uniformly accelerating frame without gravity, and (ii) an inertial reference frame containing a uniform gravitational field (See Problem 6.39). Einstein therefore saw a deep connection between *gravitation* and *accelerating reference frames*.

A second clue is the type of geometry needed within accelerating frames, as shown in the following thought experiment.

A THOUGHT EXPERIMENT

A large horizontal turntable rotates with angular velocity ω: A reference frame rotating with the turntable is an accelerating frame, because every point on it is accelerating toward the center with $a = r\omega^2$. A colony of ants living on the turntable is equipped with meter sticks to make measurements (see Figure 6.16). A *second* colony of ants lives on a *non*rotating horizontal glass sheet suspended above the turntable; these ants are also equipped with meter sticks, and can make distance measurements on the glass while they are watching beneath them the rotating ants making similar measurements directly on the turntable itself.

Both ant colonies can measure the radius and circumference of the turntable, as shown in Figure 6.16. The *inertial* ants on the glass sheet, looking down on the turntable beneath them, lay out a straight line of sticks from the turntable's center to its rim, and so find that the radius of the turntable (as measured on the glass sheet) is R_0. They also lay meter sticks end-to-end on the glass sheet around the rim of the turntable (as they see it through the glass), and find that the circumference of the turntable (according to their measurements) is $C_0 = 2\pi R_0$, verifying that Euclidean geometry is valid in their inertial frame.

The ants living on the turntable make similar measurements. Laying meter sticks along a radial line, they find that the radius is R. Laying sticks end-to-end around the rim, they find that the circumference of the turntable is C. Meanwhile the inertial ants, watching the rotating ants beneath them, find that the rotating-ant meter sticks laid out radially have no Lorentz contraction relative to their own

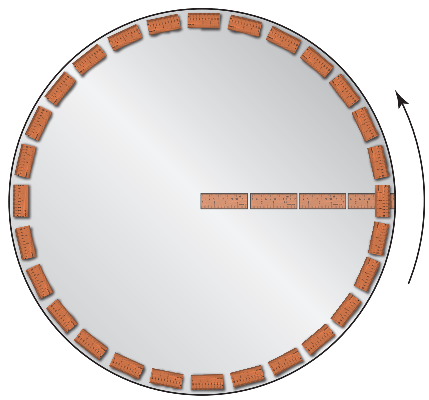

Figure 6.16: An ant colony measures the radius and circumference of a turntable.

inertial sticks, because the rotating sticks at each instant move *sideways* rather than *lengthwise*. So the inertial ants see that the rotating ants require exactly the same number of radial sticks as the inertial ants do themselves; in other words both sets of ants measure the same turntable radius, $R = R_0$.

However, the meter sticks laid out around the turntable *rim* by the rotating ants are moving with speed $v = R_0\omega$ in the direction of their lengths, and so will be Lorentz-contracted as observed in the inertial frame, so more of these meter sticks will be needed by the rotating ants to go around the rim than is required by the inertial ants. Therefore it must be that the circumference C measured by the rotating ants is *greater* than the circumference C_0 measured by the inertial ants. Since the measured radius is the same, this means that in the accelerating frame $C > 2\pi R$. The logical deduction that $C > 2\pi R$ in the accelerating frame means that the geometry actually measured in the rotating frame is **non-Euclidean**, since the measurements are in conflict with Euclidean geometry. So this thought experiment shows that there appears to be a connection between *accelerating frames* and **non-Euclidean geometry**.

A third clue to a relativistic theory of gravity is that people have long known that Euclidean geometry is the geometry on a plane, while non-Euclidean geometries are the geometries on curved surfaces. Draw a circle on the curved surface of the Earth, for example, such as a constant-latitude line in the northern hemisphere (see Figure 6.17). Then the North Pole is at the center of the circle. The radius

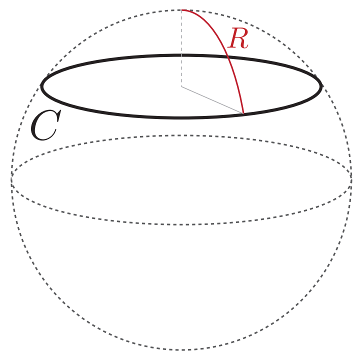

Figure 6.17: Non-Euclidean geometry: circumferences on a sphere.

of the circle is a line on the sphere extending from the center to the circle itself; in the case of the Earth, this is a line of constant longitude. Then it is easy to show that the circumference and radius obey $C < 2\pi R$. The geometry on a two-dimensional curved space like the surface of the Earth is non-Euclidean. The opposite relationship holds if a circle is drawn on a saddle, with the center of the circle in the middle of the saddle; in that case one can show that $C > 2\pi R$, so that the geometry on a curved saddle is also non-Euclidean.

This suggests the question: *Are gravitational fields* associated with *curved spaces*? Let us emphasize how special is the gravitational force in this aspect. If we were to write Newton's second law for a probe of gravitational *and* inertial mass m near a larger mass M, we would get

$$m\,\mathbf{a} = -G\frac{M\,m}{r}\hat{\mathbf{r}} \tag{6.78}$$

as we already know. Because the m's on both sides of this equation are the same, they cancel, and we get

$$\mathbf{a} = -G\frac{M}{r}\hat{\mathbf{r}}\ ; \tag{6.79}$$

that is, all objects fall in gravity with the *same* acceleration that depends only on the source mass M and its location. An elephant experiences the same gravitational acceleration as does a feather. Contrast this with the Coulombic force:

$$m\,\mathbf{a} = \frac{1}{4\pi\varepsilon_0}\frac{Q\,q}{r}\hat{\mathbf{r}} \tag{6.80}$$

where the probe has mass m and charge q, and the source has charge Q. We see that m does not drop out, and the acceleration of a probe under the influence of the electrostatic force depends on the probe's attributes: its mass and charge. This dependence of a probes acceleration on its physical attributes is generic of all force laws except gravity! The gravitational force is very special in that it has a universal character — independent of the attributes of the object it acts on. Hence, gravity lends itself to be tucked into the fabric of space itself: all probes gravitate in the same way, and thus perhaps we can think of gravity as an attribute of space itself!

The next thought experiment is an illustration how this distortion of space due to gravity affect time as well! so that, in four-dimensional language, gravity can be packaged in *spacetime.*

ANOTHER THOUGHT EXPERIMENT

Two clocks A and B are at rest in a uniform gravitational field g, with A on the ground and B at altitude h directly above. At time $t = t_0$ on A, A sends a light signal up to B, arriving at B at time t'_0 according to B, as shown in Figure 6.18. Later, when A reads t_1, A sends a second light signal to B, which arrives at t'_1 according to B. The time interval on A is $\Delta t_A = t_1 - t_0$, and the time interval on B is $\Delta t_B = t'_1 - t'_0$, which is greater than Δt_A, because from the principle of equivalence presented at the end of Chapter 3, high-altitude clocks run fast compared with low-altitude clocks. Now notice that the light signals together with the two clock world lines form a parallelogram in spacetime. Two of the sides are the parallel horizontal timelike world lines of the clocks, and the other two are the slanted null lines, corresponding to light signals. In Euclidean geometry, opposite sides of a parallelogram have the same length. But in the spacetime parallelogram, the two parallel horizontal lines, which are the clock world lines, have timelike "lengths" (measured by clock readings) $c\Delta t_A$ and $c\Delta t_B$, which are not equal (assuming c remains unchanged). Therefore when gravity is added to spacetime, the geometry becomes non-Euclidean, and so the spacetime has in some sense become *curved.* If there were no gravity the high and low clocks would tick at the same rate, so the two clock worldines would have the same timelike length, and the parallelogram would obey the rules of Euclidean geometry, corresponding to a *flat*, Euclidean spacetime. *Gravity has curved the spacetime.*

Einstein knew that physics takes place in the arena of four-dimensional spacetime. From the clues that (i) *gravity is related to accelerating reference frames*, (ii) *accelerating reference frames are related to non-Euclidean geometries*, and (iii) *non-Euclidean geometries are related to curved spaces*, Einstein became convinced

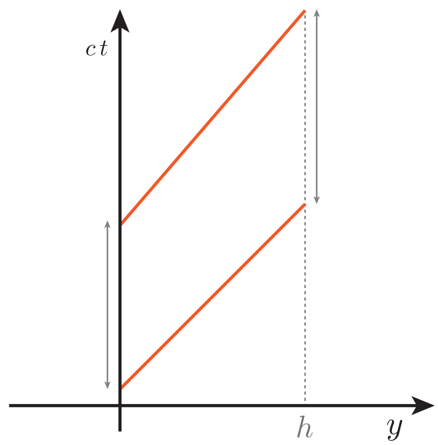

Figure 6.18: Successive light rays sent to a clock at altitude h from a clock on the ground.

that gravity is an effect of curved four-dimensional spacetime. His quest to see how gravity is related to geometry ultimately led him to the famous Einstein Field Equations of 1915. These equations showed how the geometry was affected by the particles and fields within spacetime. Then given the geometry, particles that are affected only by gravity (that is, only by the curvature of spacetime) move on the straightest lines possible in the curved spacetime, *i.e.*, along geodesics in four dimensions. General relativity has been summarized in a nutshell by the American physicist John Archibald Wheeler,

Matter tells space how to curve; space tells matter how to move.

Here we will simply present the solution of Einstein's equations for the curved spacetime surrounding a central spherically-symmetric mass M like the Sun. The solution takes the form of a spacetime **metric** analogous to the Minkowski metric of special relativity corresponding to flat spacetime. In spherical coordinates (r, θ, φ), the flat Minkowski metric is

$$ds^2 = -c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{6.81}$$

Remember that one can use this to measure Euclidean distance. For example, to compute the distance between two points measured simultaneously, we take $dt = 0$ and integrate $l = \int ds$ along the line joining the two points. Consider the setup of a probe of mass m in the vicinity of a source mass $M \gg m$. In our Newtonian language, we have the reduced mass $\mu \simeq m$ and we focus on the dynamics of the probe. The curved spherically-symmetric metric around a mass M is the **Schwarzschild metric**

$$ds^2 = -(1 - 2\mathcal{M}/r)c^2 dt^2 + (1 - 2\mathcal{M}/r)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (6.82)$$

where $\mathcal{M} \equiv GM/c^2$. Note that as $M \to 0$ the spacetime becomes flat.

In general relativity a particle subject to nothing but gravity moves on **geodesics** in four-dimensional spacetime, analogous to the geodesics on two-dimensional curved surfaces like the spherical surfaces discussed in Chapter 3. The interval obeys $ds^2 = -c^2 d\tau^2$ along a timelike worldline, just as in flat spacetime as described in Chapter 2, and timelike geodesics in the four-dimensional Schwarzschild geometry are found by making stationary the proper time along the path,

$$I = c \int d\tau. \quad (6.83)$$

From spherical symmetry we know that the geodesics will be in a plane, which (as usual) we take to be the equatorial plane $\theta = \pi/2$, leaving the degrees of freedom r and φ. Therefore we seek to find the paths that make stationary the integral

$$\begin{aligned} I &= c \int d\tau = \int \sqrt{(1 - 2\mathcal{M}/r)c^2 dt^2 - (1 - 2\mathcal{M}/r)^{-1} dr^2 - r^2 d\varphi^2} \\ &= \int \sqrt{(1 - 2\mathcal{M}/r)c^2 \dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1} \dot{r}^2 - r^2 d\dot{\varphi}^2}\, d\tau \end{aligned} \quad (6.84)$$

where $\dot{t} = dt/d\tau$, $\dot{r} = dr/d\tau$, etc. Note that while in non-relativistic physics the time t is an independent variable, and not a coordinate, in relativistic physics the time t has become one of the coordinates, and the independent variable is the proper time τ read by a clock carried along with the moving particle.

This calculus of variations problem looks exactly like the principle of stationary action, with τ replacing t as the independent variable, and a Lagrangian that is the square-root integrand. There is a way to simplify this form before making calculations. Note first of all that (since $ds^2 = -c^2 d\tau^2$)

$$(1 - 2\mathcal{M}/r)c^2 \dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1} \dot{r}^2 - r^2 d\dot{\varphi}^2 = c^2, \quad (6.85)$$

a constant along the world-line of the particle. This fact can be used to help show that making stationary the integral in equation (6.84) is equivalent to making stationary the same integral, with the same integrand but with the square root removed (See Problem 6.44). So our principle of stationary action becomes

$$\delta S = \delta \int L\, d\tau = 0 \quad (6.86)$$

where the effective Lagrangian is

$$L = (1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)\dot{r}^2 - r^2\dot{\varphi}^2. \tag{6.87}$$

Of the three coordinates, t and φ are cyclic, so the corresponding generalized momenta $p = \partial L/\partial \dot{q}$ are conserved, giving us two first integrals of motion,

$$\begin{aligned} p^t &= \frac{\partial L}{\partial \dot{t}} = -2c^2(1 - 2\mathcal{M}/r)\dot{t} \equiv -2c^2\mathcal{E} \\ p^\varphi &= \frac{\partial L}{\partial \dot{\varphi}} = 2r^2\dot{\varphi} \equiv 2\mathcal{L} \end{aligned} \tag{6.88}$$

where $\mathcal{E}$ and $\mathcal{L}$ are constants. For the third first-integral we see that L is not an explicit function of the independent variable τ, so the analog of the Hamiltonian is conserved. However, it turns out that this is equivalent to equation (6.85), already a first integral of motion. Then we can eliminate $\dot{t}$ and $\dot{\varphi}$ in equation (6.85) using the two other first integrals, giving

$$\dot{r}^2 - \frac{2\mathcal{M}c^2}{r} + \frac{\mathcal{L}^2}{r^2}\left(1 - \frac{2\mathcal{M}}{r}\right) = c^2(\mathcal{E}^2 - 1). \tag{6.89}$$

This looks more familiar if we divide by two, multiply by the mass m of the orbiting particle, recall that $\mathcal{M} \equiv GM/c^2$, and let $\mathcal{L} \equiv \ell/m$:

$$\frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{\ell^2}{2mr^2}\left(1 - \frac{2GM}{rc^2}\right) = \frac{mc^2}{2}(\mathcal{E}^2 - 1) \equiv E \tag{6.90}$$

which has the form of a one-dimensional conservation of energy equation (!)

$$E = \frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) \tag{6.91}$$

where

$$U_{\text{eff}}(r) \equiv -\frac{GMm}{r} + \frac{\ell^2}{2mr^2}\left(1 - \frac{2GM}{rc^2}\right). \tag{6.92}$$

The first two terms of $U_{\text{eff}}(r)$, which are by far the largest, are *exactly the same as the corresponding effective potential for Newtonian gravity* (6.27) with $\mu \simeq m$![12] Note then that Einstein gravity's predictions for our probe's orbit can be approximated with those of Newtonian gravity when

$$\frac{GM}{rc^2} \qquad 1\ ; \tag{6.93}$$

[12]If this were not true, the theory would be dead in the water, because we know that Newtonian gravitation is extremely accurate, at least within the solar system.

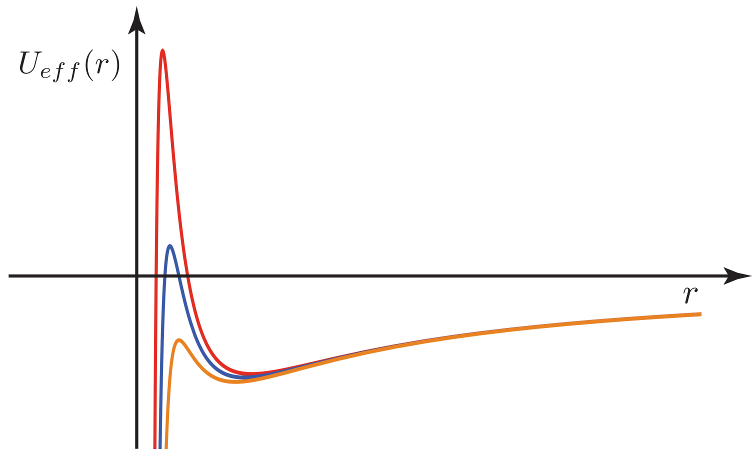

Figure 6.19: Effective potential for the Schwarzschild geometry.

that is for small source masses, or large distances from the source. In short, for "weak gravity". In fact, Einstein gravity can become important in other regimes as well, ones involving even weak gravity under the right circumstances.

In Newtonian mechanics the first term $-GMm/r$ in equation (6.92) is the gravitational potential and the second term $\ell^2/2mr^2$ is the centrifugal potential. The *third* term is new and obviously relativistic, since it involves the speed of light. It has the effect of diminishing the centrifugal potential for small r, and can make the centrifugal term *attractive* rather than *repulsive*, as shown in Figure 6.19. This effect cannot be seen for the Sun or most stars, however, because their radii are larger than the radius at which the effective potential turns around and takes a nosedive at small r, and $U_{\text{eff}}(r)$ is only valid in the vacuum *outside* the central mass. The relativistic term can have a small but observable effect on the inner planets, however, as we will show in the next example.

EXAMPLE 6-5: The precession of Mercury's perihelion

By the end of the nineteenth century astronomers knew there was a problem with the orbit of Mercury. In the Sun's inertial frame, the perihelion of Mercury's orbit does not keep returning to the same spot. The perihelion slowly *precesses*, so that each time Mercury orbits the Sun the perihelion occurs slightly later than it did on the previous revolution. The main reason for this is that the other planets pull slightly on Mercury, so the force it experiences is not purely central. Very accurate methods were worked out to calculate the total precession of Mercury's perihelion caused by the other planets, and although the calculations explained most of the precession, Mercury actually precesses by 43 seconds of arc per century more than the calculations predicted. Einstein was aware of this discrepancy when he worked on his general theory, and was intensely curious whether the effects of relativity might explain the 43 seconds/century drift.

We have already shown that the conservation equations we derived from the geodesics of the Schwarzschild geometry differ slightly from those for the nonrelativistic Kepler problem. Could the extra term in the effective potential cause the precession?

Begin with the conservation equations

$$E = \frac{1}{2}m\dot{r}^2 - \frac{GMm}{r} + \frac{\ell^2}{2mr^2}\left(1 - \frac{2GM}{rc^2}\right) \quad \text{and} \quad \mathcal{L} = \ell/m = r^2\dot{\varphi}. \tag{6.94}$$

Using the chain rule, and again defining the inverse radius $u = 1/r$ as coordinate,

$$\dot{r} \equiv \frac{dr}{d\tau} = \frac{dr}{du}\frac{du}{d\varphi}\frac{d\varphi}{d\tau} = -\frac{1}{u^2}\,u'\,\frac{\ell}{mr^2} = -\frac{\ell}{m}u' \tag{6.95}$$

where $u' \equiv du/d\varphi$. Substituting this result into equation (6.94) gives

$$E = \frac{\ell^2}{2m}(u'^2 + u^2) - GMmu - \frac{GM\ell^2}{mc^2}u^3. \tag{6.96}$$

Then differentiating with respect to φ, we get a second-order differential equation for the orbital shape $u(\varphi)$,

$$u'' + u = \frac{GMm^2}{\ell^2} + \frac{3GM}{c^2}u^2, \tag{6.97}$$

which is the same equation we found for the nonrelativistic Kepler problem, except for the second term on the right, which makes the equation nonlinear. We don't have to solve the equation exactly, however, because the new term is very small. We can solve the problem to sufficient accuracy using what is called first-order perturbation theory. Let $u = u_0 + u_1$, where u_0 is the solution of the linear equation without the new relativistic term, and u_1 is a new (small) contribution due to the relativistic term. Our goal is to find the small function u_1, to see whether the corrected solution leads to a precession of Mercury's orbit. Substituting $u = u_0 + u_1$ into equation (6.97),

$$u_0'' + u_0 + (u_1'' + u_1) = \frac{GMm^2}{\ell^2} + \frac{3GM}{c^2}(u_0 + u_1)^2. \tag{6.98}$$

The function u_0 already obeys the nonrelativistic Kepler equation

$$u_0'' + u_0 = \frac{GMm^2}{\ell^2}, \tag{6.99}$$

so the part left over is

$$u_1'' + u_1 = \frac{3GM}{c^2}(u_0 + u_1)^2. \tag{6.100}$$

Even the quantity $(3GM/c^2)u_0^2$ is already very small, so we neglect the u_1 function on the right, which would produce a doubly-small term. That leaves

$$u_1'' + u_1 = \frac{3GM}{c^2}u_0^2 \tag{6.101}$$

to first-order accuracy, which is a *linear* differential equation. We already know u_0 from the Kepler problem:

$$u_0 = A + B\cos\varphi = A(1 + e\cos\varphi) \tag{6.102}$$

where ϵ is the eccentricity of the orbit and $A \equiv [a(1 - e^2)]^{-1}$. Therefore the u_1 equation becomes

$$u_1'' + u_1 = \frac{3GM}{c^2}A^2(1 + e\cos\varphi)^2 = C^2[1 + 2e\cos\varphi + \frac{e^2}{2}(1 + \cos 2\varphi)] \tag{6.103}$$

where we have set $C^2 \equiv (3GM/c^2)A^2$ and used the identity $\cos^2\varphi = (1/2)(1 + \cos 2\varphi)$. This gives us three linearly independent terms on the right,

$$u_1'' + u_1 = C^2[(1 + e^2/2) + 2e\cos\varphi + \frac{e^2}{2}\cos 2\varphi.] \tag{6.104}$$

Note that this equation is *linear*, and that its general solution is the sum of the solution of the homogeneous equation (with zero on the right), and a particular solution of the full equation. We do not need the solution of the homogeneous equation, however, because it is the same as that for the u_0 equation, so contributes nothing new. And (because of the linearity of the equation) the particular solution of the full equation is just the sum of the particular solutions due to each of the three terms on the right, taken one at a time. That is, $u_1 = u_1^{(1)} + u_1^{(2)} + u_1^{(3)}$, where[13]

$$\begin{aligned}
u_1'' + u_1 &= C^2(1 + e^2/2), \quad &&\text{with solution } u_1^{(1)} = C^2(1 + e^2/2) \\
u_1'' + u_1 &= C^2(e^2/2)\cos 2\varphi, \quad &&\text{with solution } u_1^{(3)} = -(e^2C^2/6)\cos 2\varphi \\
u_1'' + u_1 &= 2C^2 e\cos\varphi, \quad &&\text{with solution } u_1^{(2)} = eC^2\varphi\sin\varphi
\end{aligned} \tag{6.105}$$

[13]See Problem 6.40 to work out the solutions.

Altogether, the new contribution to the solution in first order is

$$u_1 = C^2[1 + e^2/2 - (e^2/6)\cos 2\varphi + e\varphi \sin\varphi]. \tag{6.106}$$

The only term here that can cause a perihelion precession is the $\varphi \sin\varphi$ term, the so-called **secular term**, since it is the only term that does not return to where it began after a complete revolution, i.e., as $\varphi \to \varphi + 2\pi$. The other terms can cause a slight change in shape, but not a precession. So including the secular term together with the zeroth-order terms,

$$u = u_0 + u_1 = A(1 + e\cos\varphi) + C^2 e\varphi \sin\varphi. \tag{6.107}$$

The perihelion corresponds to the minimum value of r, or the maximum value of u, so at perihelion,

$$\frac{du}{d\varphi} = 0 = -Ae\sin\varphi + C^2 e(\sin\varphi + \varphi\cos\varphi), \tag{6.108}$$

which has a solution at $\varphi = 0$, but *not* at $\varphi = 2\pi$. So we look for a solution at $\varphi = 2\pi + \delta$ for some small δ. For small δ, $\sin(2\pi + \delta) = \sin\delta \cong \delta$ and $\cos(2\pi + \delta) = \cos\delta \cong 1$ to first order in δ. Therefore at the end of one revolution, equation (6.108) gives

$$0 = -Ae\delta + C^2 e[\delta + (2\pi + \delta)]. \tag{6.109}$$

However, the δ's in the square brackets are small compared with 2π, so must be neglected for consistency, since C^2 is already very small. Thus we find that

$$\delta = 2\pi C^2/A = 2\pi\left(\frac{3GM}{c^2}\right)\frac{1}{a(1-e^2)} = \frac{6\pi GM}{c^2 a(1-e^2)}. \tag{6.110}$$

The data for Mercury's orbit is $a = 5.8 \times 10^{10}$ m, $\epsilon = 0.2056$, and $M = M_{\text{Sun}} = 2.0 \times 10^{30}$ kg. The result is

$$\delta = 5.04 \times 10^{-7} \text{ radians/revolution}. \tag{6.111}$$

We can convert this result to seconds of arc/century, using the facts that Mercury orbits the Sun every 88 days and that there are $60 \times 60 = 3600$ seconds of arc in one degree,

$$\begin{aligned} \delta &= 5.04 \times 10^{-7}\,\frac{\text{rad}}{\text{rev}}\left(\frac{360\text{ deg}}{2\pi\text{ rad}}\right)\left(\frac{3600\text{ s}}{\text{deg}}\right)\left(\frac{1\text{ rev}}{88\text{ d}}\right)\left(\frac{365\text{ d}}{\text{yr}}\right)\left(\frac{100\text{ yr}}{\text{cent}}\right) \\ &= 43\,\frac{\text{seconds of arc}}{\text{century}}\ ! \end{aligned} \tag{6.112}$$

After the extraordinary efforts and frequent frustrations leading up to his discovery of general relativity, here was Einstein's payoff. He had successfully explained a well-known and long standing conundrum. Later he wrote to a friend, "For a few days, I was beside myself with joyous excitement." And in the words of his biographer Abraham Pais, "This discovery was, I believe, by far the strongest emotional experience in Einstein's scientific life, perhaps in all his life. Nature had spoken to him. He had to be right."

6.8 Exercises and Problems

PROBLEM 6-1 : Two satellites of equal mass are each in circular orbits around the Earth. The orbit of satellite A has radius r_A, and the orbit of satellite B has radius $r_B = 2r_A$. Find the ratio of their (a) speeds (b) periods (c) kinetic energies (d) potential energies (e) total energies.

PROBLEM 6-2 : Halley's comet passes through Earth's orbit every 76 years. Make a close estimate of the maximum distance Halley's comet gets from the Sun.

PROBLEM 6-3 : Two astronauts are in the same circular orbit of radius R around the Earth, 180^o apart. Astronaut A has two cheese sandwiches, while Astronaut B has none. How can A throw a cheese sandwich to B? In terms of the astronaut's period of rotation about the Earth, how long does it take the sandwich to arrive at B? What is the semimajor axis of the sandwich's orbit? (There are many solutions to this problem, assuming that A can throw the sandwich with arbitrary speeds.)

PROBLEM 6-4 : Suppose that the gravitational force exerted by the Sun on the planets were inverse r - squared, but not proportional to the planet masses. Would Kepler's third law still be valid in this case?

PROBLEM 6-5 : Planets in a hypothetical solar system all move in circular orbits, and the ratio of the periods of any two orbits is equal to the ratio of their orbital radii *squared*. How does the central force depend on the distance from this Sun?

PROBLEM 6-6 : An astronaut is marooned in a powerless spaceship in circular orbit around the asteroid *Vesta*. The astronaut reasons that puncturing a small hole through the spaceship's outer surface into an internal water tank will lead to a jet action of escaping water vapor expanding into space. Which way should the jet be aimed so the spacecraft will descend in the least time to the surface of Vesta? (In Isaac Asimov's first published story *Marooned off Vesta*, the jet was not oriented in the optimal way, but the ship reached the surface anyway.)

PROBLEM 6-7 : A thrown baseball travels in a small piece or an elliptical orbit before it strikes the ground. What is the semimajor axis of the ellipse? (Neglect air resistance.)

PROBLEM 6-8 : Assume that the period of elliptical orbits around the Sun depends only upon G, M (the Sun's mass), and a, the semimajor axis of the orbit. Prove Kepler's third law using dimensional arguments alone.

PROBLEM 6-9 : A spy satellite designed to peer closely at a particular house every day at noon has a 24-hour period, and a perigee of 100 km directly above the house. (a) What is the altitude of the satellite at apogee? (c) What is the speed of the satellite at perigee? (Earth's radius is 6400 km.)

PROBLEM 6-10 : Show that the kinetic energy

$$T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2$$

of a system of two particles can be written in terms of their center-of-mass velocity $\dot{\mathbf{R}}_{cm}$ and relative velocity $\dot{\mathbf{r}}$ as

$$T = \frac{1}{2}M\dot{\mathbf{R}}_{cm}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$$

where $M = m_1 + m_2$ is the total mass and $\mu = m_1 m_2/M$ is the reduced mass of the system.

PROBLEM 6-11 : Show that the shape $r(\theta)$ for a central spring force ellipse takes the standard form $r^2 = a^2b^2/(b^2\cos^2\theta + a^2\sin^2\theta)$ if (in equation (6.37)) we use the plus sign in the denominator and choose $\theta_0 = \pi/4$.

PROBLEM 6-12 : Show that the period of a particle that moves in a circular orbit close to the surface of a sphere depends only upon G and the average density ρ of the sphere. Find what this period would be for *any* sphere having an average density equal to that of water. (The sphere consisting of the planet Jupiter nearly qualifies!)

PROBLEM 6-13 : (a) Communication satellites are placed into geosynchronous orbits; that is, they typically orbit in Earth's equatorial plane, with a period of 24 hours. What is the radius of this orbit, and what is the altitude of the satellite above Earth's surface? (b) A satellite is to be placed in a synchronous orbit around the planet Jupiter to study the famous "red spot". What is the altitude of this orbit above the "surface" of Jupiter? (The rotation period of Jupiter is 9.9 hours, its mass is about 320 Earth masses, and its radius is about 11 times that of Earth.)

PROBLEM 6-14 : The perihelion and aphelion of the asteroid *Apollo* are 0.964×10^8 km and 3.473×10^8 km from the Sun, respectively. Apollo therefore swings in and out through Earth's orbit. Find (a) the semimajor axis (b) the period of Apollo's orbit, given the Sun's mass $M = 1.99 \times 10^{30}$ kg. (Apollo is only one of many "Apollo asteroids" that cross Earth's orbit. Some have struck the Earth in the past, and others will strike in the future unless we find a way to prevent it.)

PROBLEM 6-15 : (a) Evaluate the integral in equation (6.29) to find $t(r)$ for a particle moving in a central gravitational field. (b) From the results, derive the equation for the period $T = (2\pi/\sqrt{GM})a^{3/2}$ in terms of the semimajor axis a for particles moving in elliptical orbits around a central mass.

PROBLEM 6-16 : The Sun moves at a speed $v_S = 220$ km/s in a circular orbit of radius $r_S = 30,000$ light years around the center of the Milky Way galaxy. The Earth requires $T_E = 1$ year to orbit the Sun, at a radius of 1.50×10^{11} m. (a) Using this information, find a formula for the total mass responsible for keeping the Sun in its orbit, as a multiple of the Sun's mass M_0, in terms also of the parameters v_S, r_S, T_E, and r_E. Note that G is not needed here! (b) Find this mass numerically.

PROBLEM 6-17 : The two stars in a double-star system circle one another gravitationally, with period T. If they are suddenly stopped in their orbits and allowed to fall together, show that they will collide after a time $T/4\sqrt{2}$.

PROBLEM 6-18 : A particle is subjected to an attractive central spring force $F = -kr$. Show, *using Cartesian coordinates*, that the particle moves in an elliptical orbit, with the force center at the *center* of the eilipse, rather than at one focus of the ellipse.

PROBLEM 6-19 : Use equation (6.32) to show that if the central force on a particle is $F = 0$, the particle moves in a straight line.

PROBLEM 6-20 : Find the central force law $F(r)$ for which a particle can move in a spiral orbit $r = k\theta^2$, where k is a constant.

PROBLEM 6-21 : Find two second integrals of motion in the case $F(r) = -k/r^3$, where k is a constant. Describe the shape of the trajectories.

PROBLEM 6-22 : A particle of mass m is subject to a central force $F(r) = -GMm/r^2 - k/r^3$, where k is a positive constant. That is, the particle experiences an inverse-cubed attractive force as well as a gravitational force. Show that if k is less than some limiting value, the motion is that of a precessing ellipse. What is this limiting value, in terms of m and the particle's angular momentum?

PROBLEM 6-23 : Find the allowed orbital shapes for a particle moving in a *repulsive* inverse-square central force. These shapes would apply to α - particles scattered by gold nuclei, for example, due to the repulsive Coulomb force between them.

PROBLEM 6-24 : A particle moves in the field of a central force for which the potential energy is $U(r) = kr^n$, where both k and n are constants, positive, negative, or zero. For what range of k and n can the particle move in a stable, circular orbit at some radius?

PROBLEM 6-25 : A particle of mass m and angular momentum ℓ moves in a central spring-like force field $F = -kr$. (a) Sketch the effective potential energy $U_{\text{eff}}(r)$. (b) Find the radius r_0 of circular orbits. (c) Find the period of small oscillations about this orbit, if the particle is perturbed slightly from it. (d) Compare with the period of rotation of the particle about the center of force. Is the orbit closed or open for such small oscillations?

PROBLEM 6-26 : Find the period of small oscillations about a circular orbit for a planet of mass m and angular momentum ℓ around the Sun. Compare with the period of the circular orbit itself. Is the orbit open or closed for such small oscillations?

PROBLEM 6-27 : (a) A binary star system consists of two stars of masses m_1 and m_2 orbiting about one another. Suppose that the orbits of the two stars are circles of radii r_1 and r_2, centered on their center of mass. Show that the period of the orbital motion is given by

$$T^2 = \frac{4\pi^2}{G(m_1 + m_2)}(r_1 + r_2)^2.$$

(b) The binary system Cygnus X-1 consists of two stars orbiting about their common center of mass with orbital period 5.6 days. One of the stars is a supergiant with a mass 25 times that of the Sun. The other star is believed to be a black hole with a mass of about 10 times the mass of the Sun. From the information given, determine the distance between these stars, assuming that the orbits are circular.

PROBLEM 6-28 : A spacecraft is in a circular orbit of radius r about the Earth. What is the minimum Δv (in km/s) the rocket engines must provide to allow the craft to escape from the Earth?

PROBLEM 6-29 : A spacecraft is designed to dispose of nuclear waste either by carrying it out of the solar system or by crashing it into the Sun. Which mission requires the least rocket fuel? (Do not include possible gravitational boosts from other planets or worry about escaping from Earth's gravity.)

PROBLEM 6-30 : After the engines of a 100 kg spacecraft have been shut down, the spacecraft is found to be a distance 10^7 m from the center of the Earth, moving with a speed of 7000 m/s at an angle of 45^o relative to a straight line from the Earth to the spacecraft. (a) Calculate the total energy and angular momentum of the spacecraft. (b) Determine the semimajor axis and the eccentricity of the spacecraft's geocentric trajectory.

PROBLEM 6-31 : A 100 kg spacecraft is in circular orbit around the Earth, with orbital radius 10^4 km and with speed 6.32 km/s. It is desired to turn on the rocket engines to accelerate the spacecraft up to a speed so that it will escape the Earth and coast out to Jupiter. Use a value of 1.5×10^8 km for the radius of Earth's orbit, 7.8×10^8 km for Jupiter's orbital radius, and a value of 30 km/s for the velocity of the Earth. Determine (a) the semimajor axis of the Hohmann transfer orbit to Jupiter; (b) the travel time to Jupiter; (c) the heliocentric velocity of the spacecraft as it leaves the Earth; (d) the minimum Δv required from the engines to inject the spacecraft into the transfer orbit.

PROBLEM 6-32 : The Earth-Sun L5 Lagrange point is a point of stable equilibrium that trails the Earth in its heliocentric orbit by 60^o as the Earth (and spacecraft) orbit the Sun. Some gravity wave experimenters want to set up a gravity wave experiment at this point. The simplest trajectory from Earth puts the spacecraft on an elliptical orbit with a period slightly longer than one year, so that, when the spacecraft returns to perihelion, the L5 point will be there. (a) Show that the period of this orbit is 14 months. (b) What is the semimajor axis of this elliptical orbit? (c) What is the perihelion speed of the spacecraft in this orbit? (d) When the spacecraft finally reaches the L5 point, how much velocity will it have to lose (using its engines) to settle into a circular heliocentric orbit at the L5 point?

PROBLEM 6-33 : In *Stranger in a Strange Land*, Robert Heinlein claims that travelers to Mars spent 258 days on the journey out, the same for return, "plus 455 days waiting at Mars while the planets crawled back into positions for the return orbit." Show that travelers *would* have to wait about 455 days, if both Earth-Mars journeys were by Hohmann transfer orbits.

PROBLEM 6-34 : A spacecraft approaches Mars at the end of its Hohmann transfer orbit. (a) What is its velocity in the Sun's frame, before Mars's gravity has had an appreciable influence on it? (b) What Δv must be given to the spacecraft to insert it directly from the transfer orbit into a circular orbit of radius 6000 km around Mars?

PROBLEM 6-35 : A spacecraft parked in circular low-Earth orbit 200 km above the ground is to travel out to a circular geosynchronous orbit, of period 24 hours, where it will remain. (a) What initial Δv is required to insert the spacecraft into the transfer orbit? (b) What final Δv is required to enter the synchronous orbit from the transfer orbit?

PROBLEM 6-36 : A spacecraft is in a circular parking orbit 300 km above Earth's surface. What is the transfer-orbit travel time out to the Moon's orbit, and what are the two $\Delta v's$ needed? Neglect the Moon's gravity.

PROBLEM 6-37 : A spacecraft is sent from the Earth to Jupiter by a Hohmann transfer orbit. (a) What is the semimajor axis of the transfer ellipse? (b) How long does it take the spacecraft to reach Jupiter? (c) If the spacecraft actually leaves from a circular parking orbit around the Earth of radius 7000 km, find the rocket Δv required to insert the spacecraft into the transfer orbit.

PROBLEM 6-38 : Find the Hohmann transfer-orbit time to Venus, and the $\Delta v's$ needed to leave an Earth parking orbit of radius 7000 km and later to enter a parking orbit around Venus, also of $r = 7000$ km. Sketch the journey, showing the orbit directions and the directions in which the rocket engine must be fired.

PROBLEM 6-39 : Consider an astronaut standing on a weighing scale within a spacecraft. The scale by definition reads the normal force exerted by the scale on the astronaut (or, by Newton's third law, the force exerted on the scale by the astronaut.) By the principle of equivalence, the astronaut can't tell whether the spacecraft is (a) sitting at rest on the ground in uniform gravity g, or (b) is in gravity-free space, with uniform acceleration a numerically equal to the gravity g in case (a). Show that in one case the measured weight will be proportional to the inertial mass of the astronaut, and in the other case proportional to the astronaut's gravitational mass. So if the principle of equivalence is valid, these two types of mass must have equal magnitudes.

PROBLEM 6-40 : Verify the particular solutions given of the inhomogeneous first-order equations for the perihelion precession, as given in equation (6.96).

PROBLEM 6-41 : Find the general-relativistic precession of Earth's orbit around the Sun, in seconds of arc per century. Earth's orbital data is $a = 1.50 \times 10^6$ m and $\epsilon = 0.0167$.

PROBLEM 6-42 : The metric of flat, Minkowski spacetime in Cartesian coordinates is $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. Show that the geodesics of particles in this spacetime correspond to motion in straight lines at constant speed.

PROBLEM 6-43 : Show from the effective potential corresponding to the Schwarzschild metric that if $U_{\rm eff}$ can be used for arbitrarily small radii, there are actually *two* radii at which a particle can be in a circular orbit. The outer radius corresponds to the usual stable, circular orbit such as a planet would have around the Sun. Find the radius of the inner circular orbit, and show that it is unstable, so that if the orbiting particle deviates slightly outward from this radius it will keep moving outward, and if it deviates slightly inward it will keep moving inward.

PROBLEM 6-44 : The geodesic problem in the Schwarzschild geometry is to make stationary the integral

$$I = \int \sqrt{(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2 d\dot{\theta}^2}\; d\tau.$$

Use this integrand in the Euler-Lagrange equations to show that one obtains exactly the same differential equations in the end if the square root is removed, i. e., if we make stationary instead the integral

$$I = \int [(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2 d\dot{\theta}^2]\; d\tau.$$

You may use the fact that $(1 - 2\mathcal{M}/r)c^2\dot{t}^2 - (1 - 2\mathcal{M}/r)^{-1}\dot{r}^2 - r^2 d\dot{\theta}^2 = c^2$, a constant along the path of the particle.

PROBLEM 6-45 : Show that there are no stable circular orbits of a particle in the Schwarzschild geometry with a radius less than $6GM/c^2$.

PROBLEM 6-46 : *Bertrand's theorem* In Section 6.4, we stated a powerful theorem that asserts that the only potentials for which all bounded orbits are closed are: $U_{eff} \propto r^2$ and $U_{eff} \propto r^{-1}$. To prove this theorem, let us proceed in steps. If a potential is to have bound orbits, the effective potential must have a minimum since a bound orbit is a dip in the effective potential. The minimum is at $r = R$ given by

$$U'(R) = \frac{\ell^2}{\mu r^2} \tag{6.113}$$

as shown in equation (6.22). This corresponds to a circular orbit which is stable if

$$U''(R) + \frac{3}{R}U'(R) > 0 \tag{6.114}$$

as shown in equation (6.23). Consider perturbing this circular orbit so that we now have an r_{min} and an r_{max} about $r = R$. Define the apsidal angle $\Delta\varphi$ as the angle between the point on the perturbed orbit at r_{min} and the point at r_{max}. Assume $(R - r_{min})/R \ll 1$ and $(r_{max} - R)/R \ll 1$. Note that closed orbits require

$$\Delta\varphi = 2\pi\frac{m}{n} \tag{6.115}$$

for integer m and n and for all R.

- Show that

$$\Delta\varphi = \pi\sqrt{\frac{U'(R)}{3\,U'(R) + R\,U''(R)}} \ . \tag{6.116}$$

  Notice that the argument under the square root is always positive by virtue of the stability of the original circular orbit.

- In general, any potential $U(r)$ can be expanded in terms of positive and negative powers of r, with the possibility of a logarithmic term

$$U(r) = \sum_{n=-\infty}^{\infty} \frac{a_n}{r^n} + a\ln r \ . \tag{6.117}$$

Show that, to have the apsidal angle independent of r, we must have: $U(r) \propto r^{-\alpha}$ for $\alpha < 2$ and $\alpha \neq 0$, or $U(r) \propto \ln r$. Show that the value of $\Delta\varphi$ is then

$$\Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}} \ , \tag{6.118}$$

where the logarithmic case corresponds to $\alpha = 0$ in this equation.

- Show that if $\lim_{r\to\infty} U(r) = \infty$, we must have $\lim_{E\to\infty} \Delta\varphi = \pi/2$. This corresponds to the case $\alpha < 0$. We then must have

$$\Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}} = \frac{\pi}{2} \ , \tag{6.119}$$

or $\alpha = -2$, thus proving one of the two cases of the theorem.

- Show that for the case $0 \leq \alpha 2$, we must have $\lim_{E\to-\infty} \Delta\varphi = \pi/(2-\alpha)$. This then implies

$$\Delta\varphi = \frac{\pi}{\sqrt{2-\alpha}} = \frac{\pi}{2-\alpha} \tag{6.120}$$

which leaves only the possibility $\alpha = 1$, completing the proof of the theorem.