Chapter 5

Symmetries and Conservation Laws

At a very fundamental level, physics is about identifying patterns of order in Nature. The inception of the field starts arguably with Tycho Brahe (1546-1601) — the first modern experimental physicist — and Johannes Kepler (1571-1630) — the first modern theoretical physicist. In the 16th century, Brahe collected large amounts of astrophysical data about the location of planets and stars with groundbreaking accuracy — using an impressive telescope set up in a castle. Kepler pondered for years over Brahe's long tables of numbers until he could finally identify a pattern underlying planetary dynamics: this was summarized in the three laws of Kepler. Later on, Isaac Newton (1643-1727) referred to these achievements through his famous quote: *"If I have seen further it is only by standing on the shoulders of giants"*.

Since then, physics has always been about identifying patterns in numbers, in measurements. And a pattern is simply an indication of a repeating rule, a constant attribute in complexity, an underlying symmetry. In 1918, Emmy Noether (1882–1935) published a seminal work that clarified the deep relations between symmetries and conserved quantities in Nature. In a sense, this work organizes physics in a clear diagram and gives one a bird's eye view of the myriad of branches of the field. Noether's theorem, as it is called, can change the way one thinks about physics in general. It is profound yet simple.

While one can study Noether's theorem in the Newtonian formulation of mechanics through cumbersome methods, the subject is an excellent demonstration of the power of the new formalism we developed, Lagrangian mechanics. In this section, we use the variational principle to develop a statement of Noether's theorem. We then prove it and demonstrate its importance through examples.

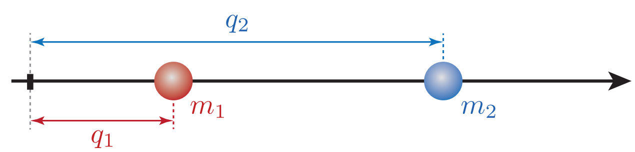

Figure 5.1: Two particles on a rail.

EXAMPLE 5-1: A simple example

Let us start with a simple mechanics problem from an example we saw in Section 4.1. We have two particles, with masses m_1 and m_2, confined to moving along a horizontal frictionless rail as depicted in Figure 5. The action for the system is

$$S = \int dt \left(\frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 - V(q_1 - q_2)\right) \tag{5.1}$$

where we are considering some interaction between the particles described by a potential $V(q_1 - q_2)$ that depends only on the distance between the particles.

Let us consider a simple **transformation** of the coordinates given by

$$q_1' = q_1 + \mathrm{C} \quad , \quad q_2' = q_2 + \mathrm{C} \tag{5.2}$$

where C is some arbitrary constant. We then have

$$\dot{q}_1' = \dot{q}_1 \quad , \quad \dot{q}_2' = \dot{q}_2 \; . \tag{5.3}$$

Hence, the kinetic terms in the action are unchanged under this transformation. Furthermore, we also have

$$q_1' - q_2' = q_1 - q_2 \tag{5.4}$$

implying that the potential term is also unchanged. The action then preserves its overall structural form under the transformation

$$S \to \int dt \left(\frac{1}{2} m_1 \dot{q'}_1^2 + \frac{1}{2} m_2 \dot{q'}_2^2 - V(q'_1 - q'_2) \right) . \tag{5.5}$$

This means that the equations of motion, written in the primed transformed coordinates, are identical to the ones written in the unprimed original coordinates. We then say that the transformation given by (5.2) is a **symmetry** of our system. Physically, we are simply saying that — since the interaction between the particles depends only on the distance between them — a constant shift of both coordinates leaves the dynamics unaffected.

It is also useful to consider an *infinitesimal* version of such a transformation. Let us assume that the constant C is small, C $\to \epsilon$; and we write

$$q'_k - q_k \equiv \delta q_k = \epsilon \tag{5.6}$$

for $k = 1, 2$. We then say that $\delta q_k = \epsilon$ is a symmetry of our system. To make these ideas more useful, we want to extend this example by considering a general class of interesting transformations that we may want to consider.

5.1 Infinitesimal transformations

There are two useful types of infinitesimal transformations: direct and indirect ones.

5.1.1 Direct transformations

A direct transformation deforms the degrees of freedom of a setup directly:

$$\delta q_k(t) = q'_k(t) - q_k(t) \equiv \Delta q_k(t, q) . \tag{5.7}$$

We use the notation Δ to distinguish a direct transformation. Note that $\Delta q_k(t, q)$ is possibly a function of time and all of the degrees of freedom in the problem. In the previous example, we had the special case where $\Delta q_k(t, q) =$ constant. But it need not be so. Figure 5.2(a) depicts a direct transformation: it is an arbitrary shift in the q_k's. Note also that $\Delta q_k(t, q)$ is a *small* deformation.

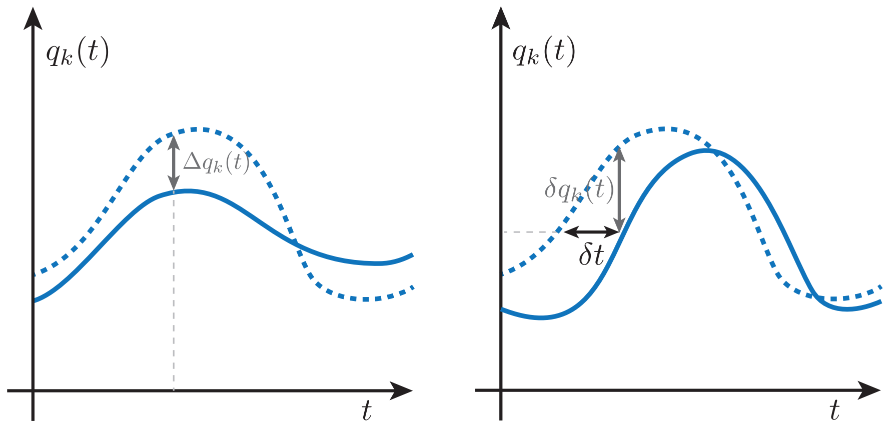

Figure 5.2: The two types of transformations considered: direct on the left, indirect on the right.

5.1.2 Indirect transformations

In contrast, an indirect transformation affects the degrees of freedom indirectly — through the transformation of the time coordinate:

$$\delta t(t, q) \equiv t' - t \ . \tag{5.8}$$

Note again that the shift in time can depend on time and the degrees of freedom as well! It is again assumed to be small. This is however not the end of the story since the degrees of freedom depend on time and will get affected as well — indirectly

$$q_k(t) = q_k(t' - \delta t) \simeq q_k(t') - \frac{dq_k}{dt'}\delta t \simeq q_k(t') - \dot{q}_k \delta t \tag{5.9}$$

where we have used a Taylor expansion in δt to linear order since δt is small. We also have used $dq_k/dt' = dq_k/dt$ since this term already multiplies a power of δt: to linear order in δt, $\delta t\, dq_k/dt' = \delta t\, dq_k/dt \equiv \delta t\, \dot{q}_k$. We then see that shifting time results in a shift in the degrees of freedom

$$\delta q_k = q_k(t') - q_k(t) = \dot{q}\, \delta t(t, q) \ . \tag{5.10}$$

Figure 5.2(b) shows how you can think of this effect graphically.

5.1.3 Combined transformations

In general, we want to consider a transformation that may include *both* direct and indirect pieces. We would write

$$\begin{aligned} \delta q_k &= {q'}_k(t') - q_k(t) = {q'}_k(t') + [-q_k(t') + q_k(t')] - q_k(t) \\ &= [{q'}_k(t') - q_k(t')] + [q_k(t') - q_k(t)] = \Delta q_k(t', q) + \dot{q}\, \delta t(t, q) \\ &= \Delta q_k(t, q) + \dot{q}\, \delta t(t, q) \ . \end{aligned} \tag{5.11}$$

where in the last line we have equated t and t' since the first term is already linear in the small parameters. To specify a particular transformation, we would then need to provide a set of functions

$$\delta t(t, q) \quad \text{and} \quad \delta q_k(t, q) \ . \tag{5.12}$$

Equation (5.11) then determines $\Delta q_k(t, q)$. For N degrees of freedom, that's $N+1$ functions of time and the q_k's. Let us look at a few examples.

EXAMPLE 5-2: Translations

Consider a single particle in three dimensions, described by the three Cartesian coordinates $x^1 = x$, $x^2 = y$, and $x^3 = z$. We also have the time coordinate $x^0 = c\,t$. An infinitesimal spatial translation can be realized by

$$\delta x^i(t, x) = \epsilon^i \quad , \quad \delta t(t, x) = 0 \Rightarrow \Delta x^i(t, x) = \epsilon^i \tag{5.13}$$

where $i = 1, 2, 3$, and the ϵ^i's are three small constants. A translation in space is then defined by

$$\{\delta t(t, x) = 0, \;\; \delta x^i(t, x) = \epsilon^i\} \tag{5.14}$$

A translation in time on the other hand would be given by

$$\delta x^i(t, x) = 0 \quad , \quad \delta t(t, x) = \varepsilon \Rightarrow \Delta x^i(t, x) = -\dot{x}^i \varepsilon \tag{5.15}$$

for constant ε. Notice that we require that the *total* shifts in the x^i's — the $\delta x^i(t, x)$'s — vanish. This then generates direct shifts, the Δx^i's, to compensate for the indirect effect on the spatial coordinates from the shifting of the time. A translation in time is then defined by

$$\{\delta t(t, x) = \epsilon, \delta x^i(t, x) = 0\} \tag{5.16}$$

EXAMPLE 5-3: Rotations

To describe rotations, let us consider a particle in two dimensions for simplicity. We use the coordinates $x^1 = x$ and $x^2 = y$. We start by specifying

$$\delta t(t,x) = 0 \Rightarrow \delta x^i(t,x) = \Delta x^i(t,x) \ . \tag{5.17}$$

Next, we look at an arbitrary rotation angle θ using (2.21)

$$\begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \ . \tag{5.18}$$

We however need to focus on an infinitesimal version of this transformation: *i.e.* we need to consider small angle θ. Using $\cos\theta \sim 1$ and $\sin\theta \sim \theta$ to second order in θ, we then write

$$\begin{pmatrix} x^{1'} \\ x^{2'} \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \ . \tag{5.19}$$

This gives

$$\begin{aligned} \delta x^1(t,x) &= x^{1'}(t) - x^1(t) = \theta\, x^2(t) = \Delta x^1(t,x) \\ \delta x^2(t,x) &= x^{2'}(t) - x^2(t) = -\theta\, x^1(t) = \Delta x^2(t,x) \ . \end{aligned} \tag{5.20}$$

We see we have a more non-trivial transformation. Rotations can then be defined through

$$\{\delta t(t,x) = 0, \delta x^i(t,x) = \theta\, \varepsilon^{ij} x^j(t)\} \tag{5.21}$$

where j is summed over 1 and 2. We have also introduced a useful shorthand: ε^{ij}. It is called the totally antisymmetric matrix in two dimensions and defined as:

$$\varepsilon^{11} = \varepsilon^{22} = 0 \quad , \quad \varepsilon^{12} = -\varepsilon^{21} = 1 \ . \tag{5.22}$$

It allows us to write the transformation in a more compact notation.

EXAMPLE 5-4: Lorentz transformations

To find the infinitesimal form of Lorentz transformations, we can start with the general transformation equations (2.15) and take β small. We need to be careful however to keep the leading order terms in β in all expansions. Given our previous example, it is easier to map the problem onto a rotation with hyperbolic trig functions using (2.27). For simplicity, let us consider a particle in one dimension, with two relevant coordinates $x^0 = c\,t$ and x^1. Looking back at (2.27), we take the rapidity ξ to be small and use $\cosh\xi \sim 1$ and $\sinh\xi \sim \xi$ to linear order in ξ. Along the same steps of the previous example, we quickly get

$$\delta x^0(t,x) = \xi\, x^1 \quad , \quad \delta x^1(t,x) = \xi\, x^0 \ . \tag{5.23}$$

Using equation (5.11), we then have

$$\Delta x^1(t,x) = \delta x^1(t,x) - \dot{x}^1 \delta t(t,x) = \xi\, c\, t - \frac{\xi}{c}\dot{x}^1 x^1 \ . \tag{5.24}$$

Note that we have

$$\sinh\xi = \gamma\beta \Rightarrow \sinh\xi \sim \xi \sim \beta \ . \tag{5.25}$$

Lorentz transformations can then be written through

$$\{\delta x^0(t,x) = \beta\, x^1 \quad , \quad \delta x^1(t,x) = \beta\, x^0 \quad , \quad \delta x^2(t,x) = \delta x^3(t,x) = 0\} \tag{5.26}$$

where we added the transverse directions to the game as well.

5.2 Symmetry

At the beginning of this section, we defined symmetry as a transformation that leaves the action unchanged in form. In that particular example of two interacting particles on a wire, it was simple to see that the transformation was indeed a symmetry. Now that we have a general class of transformations, we want to find a general condition that can be used to test whether a particular transformation, possibly a complicated one, is a symmetry or not. We then need to look at how the action changes under a general transformation; for a symmetry, this change should vanish. We start with the usual form for the action

$$S = \int dt\, L(q, \dot{q}, t) \ . \tag{5.27}$$

And we apply a general transformation given by $\Delta q_k(t,q)$ and $\delta t(t,q)$. We then have

$$\delta S = \int \delta(dt)\, L + \int dt\, \delta(L) \ , \tag{5.28}$$

where we used the Leibniz rule of derivation since δ is an infinitesimal change. The first term is the change in the measure of the integrand

$$\delta(dt) = dt\, \frac{\delta(dt)}{dt} = dt\, \frac{d}{dt}(\delta t) \ . \tag{5.29}$$

In the last bit, we have exchanged the order of derivations since derivations commute. The second term has two pieces

$$\delta(L) = \Delta(L) + \delta t \, \frac{dL}{dt} \ . \tag{5.30}$$

The first piece is the change in L resulting from its dependence on the q_k's and $\dot{q}_k$'s. Hence, we labeled it as a direct change with a Δ. The second piece is the change in L to linear order in δt due to the change in t. This comes from changes in the q_k's on which L depends, as well as changes in t directly since t can make an explicit appearance in L. This is an identical situation to the linear expansion encountered for q_k in (5.11): there's a piece from direct changes in the degrees of freedom, plus a piece from the transformation of time. We can further write

$$\Delta(L) = \frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\Delta(\dot{q}_k) = \frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) \ . \tag{5.31}$$

Note that in the last term, we exchanged the orders of derivations, Δ and d/dt: derivations are commutative. We can now put everything together and write

$$\begin{aligned} \delta S &= \int dt \left(\frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) + \delta t \frac{dL}{dt} + L\,\frac{d}{dt}(\delta t) \right) \\ &= \int dt \left(\frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) + \frac{d}{dt}(L\,\delta t) \right) \ . \end{aligned} \tag{5.32}$$

Given L, $\delta t(t,q)$, and $\Delta q_k(t,q)$, we can substitute these in (5.32) and check whether the expression vanishes. If it does vanish, we conclude that the given transformation $\{\delta t(t,q), \Delta q_k(t,q)\}$ is a symmetry of our system. This shall be our notion of symmetry. A bit later, we will revisit this statement and generalize it further. For now, this is enough to move onto the heart of the topic, Noether's theorem.

5.2.1 Noether's theorem

The theorem

We start by simply stating the theorem:

For every symmetry $\{\delta t(t,q), \Delta q_k(t,q\}$, there exists a quantity that is conserved under time evolution.

A symmetry implies a conservation law. This is important for two reasons. (1) First, a conservation law identifies a rule in the laws of Physics. Virtually everything we have a name for physics — mass, momentum, energy, charge, etc.. — is tied by definition to a conservation law. Noether's theorem then states that fundamental physics is founded on the principle of identifying the symmetries of

Nature. If one wants to know all the laws of physics, one needs to ask: what are all the symmetries in Nature. From there, you find conservation laws and associated interesting conserved quantities. You then can study how these conservation laws can be violated. This leads you to equations that you can use to predict the future with. It's all about symmetries. (2) Second, conservation laws have the form

$$\frac{d}{dt}\left(\text{Something}\right) = 0 \Rightarrow \text{Something} = \text{Constant} \ . \tag{5.33}$$

The 'Something' is typically a function of the degrees of freedom and the first derivatives of the degrees of freedom. The conservation statement then inherently leads to first order differential equations. First order differential equations are much nicer than second or higher order ones. Thus, technically, finding the symmetries and corresponding conservation laws in a problem helps a great deal in solving and understanding the physical system.

The easiest way to understand Noether's theorem is to prove it, which is a surprisingly simple exercise.

Proof of Noether's theorem

The premise of the theorem is that we have a given symmetry $\{\delta t(t,q), \Delta q_k(t,q\}$. This then implies, using (5.32), that we have

$$\delta S = 0 = \int dt\, \left(\frac{\partial L}{\partial q_k}\Delta q_k + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\Delta q_k\right) + \frac{d}{dt}(L\,\delta t)\right) \ . \tag{5.34}$$

Note that we know that this equation is satisfied for *any* set of curves $q_k(t)$ by virtue of $\{\delta t(t,q), \Delta q_k(t,q\}$ constituting a symmetry. And now comes the crucial step: what if the $q_k(t)$'s satisfied the equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} \ . \tag{5.35}$$

Of all possible curves $q_k(t)$, we pick the ones that satisfy the equations of motion. Given this additional statement, we can rearrange the terms in δS such that

$$0 = \int dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\,\delta t\right) \ . \tag{5.36}$$

Since the integration interval is arbitrary, we then conclude that

$$\frac{d}{dt}\left(Q\right) = 0 \tag{5.37}$$

with

$$Q \equiv \frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\,\delta t\ ; \tag{5.38}$$

We have a conserved quantity. Q is called the **Noether charge**. Note a few important points:

- We used the equations of motion to prove the conservation law. However, we did not use the equations of motion to conclude that a particular transformation is a symmetry. The symmetry exists at the level of the action for any $q_k(t)$. The conservation law exists for physical trajectories that satisfy the equations of motion.
- The proof identifies explicitly the conserved quantity through (5.38). Knowing L, $\delta t(t,q)$, and $\Delta q_k(t,q)$, this equation tells us immediately the conserved quantity associated with the given symmetry.

This proof also highlights a route to generalize the original definition of symmetry. All that was needed was to have

$$\delta S = \int dt \frac{d}{dt}(K) \tag{5.39}$$

where K is some function that you would find out by using (5.32). If K turns out to be a constant, we would get $\delta S = 0$ and we are back to the situation at hand. However, if K is non-trivial, we would get

$$\delta S = \int dt \frac{d}{dt}(K) = \int dt\, \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\Delta q_k + L\,\delta t\right)\ . \tag{5.40}$$

This now means that

$$\frac{d}{dt}(Q - K) = 0 \tag{5.41}$$

and hence the conserved quantity is $Q - K$ instead of Q. Since the interesting conceptual content of a symmetry is its associated conservation law, we want to turn the problem on its back: we want to define a symmetry through a conservation statement. So, here's a revised more general statement

$$\{\delta t(t,q), \Delta q_k(t,q\}\ \text{is a symmetry if}\ \delta S = \int dt\, \frac{dK}{dt}\ \text{for some K}\ . \tag{5.42}$$

Noether's theorem then states that for every such symmetry, there is a conserved quantity given by $Q - K$.

To summarize, here's then the general prescription:

1. Given a Lagrangian L and a candidate symmetry $\{\delta t(t,q), \Delta q_k(t,q\}$, use (5.32) to find δS. If $\delta S = \int dt\, dK/dt$ for some K you are to determine, $\{\delta t(t,q), \Delta q_k(t,q\}$ is indeed a symmetry.

2. If $\{\delta t(t,q), \Delta q_k(t,q\}$ was found to be a symmetry with some K, we can find an associated conserved quantity $Q - K$, with Q given by (5.38).

Let us look at a few examples.

EXAMPLE 5-5: Space translations and momentum

We start with the spatial translation transformation from our previous example (5.13)

$$\delta x^i(t,x) = \epsilon^i \quad , \quad \delta t(t,x) = 0 \Rightarrow \Delta x^i(t,x) = \epsilon^i \ . \tag{5.43}$$

We next need a Lagrangian to test this transformation against. Consider first

$$L = \frac{1}{2} m \dot{x}^i \dot{x}^i \ . \tag{5.44}$$

We substitute (5.43) and (5.44) into (5.32) and easily find

$$\delta S = 0 \Rightarrow K = \text{Constant}. \tag{5.45}$$

Hence, we have a symmetry at hand. To find the associated conserved charge, we use (5.38) and find

$$Q^i = m\dot{x}^i \epsilon^i \quad \text{No sum on } i. \tag{5.46}$$

We then have three charges for the three possible directions for translation. An overall additive or multiplicative constant is arbitrary since it does not affect the statement of conservation $\dot{Q}^i = 0$. Writing the conserved quantities as P^i instead, we state

$$P^i = m\dot{x}^i \tag{5.47}$$

i.e. momentum is the Noether charge associate with the symmetry of spatial translational invariance. If we have a physical system set up on a table and we notice that we can move the table by any amount in any of the three spatial directions *without* affecting the dynamics of the system, we can conclude that there is a quantity — called momentum by definition — that remains constant in time.

We can then try to find the conditions under which this symmetry, and hence conservation law, is violated. For example, we could consider a setup which results in adding a familiar potential term to the Lagrangian

$$L = \frac{1}{2} m \dot{x}^i \dot{x}^i - \frac{1}{2} k\, x^i x^i \; . \tag{5.48}$$

Using (5.32), we now get

$$\delta S = \int dt \,\left(-k\, x^i \epsilon^i\right) \neq \int \frac{d}{dt}\,(K) \; . \tag{5.49}$$

Hence, momentum is not conserved and we would write

$$\dot{P}^i \neq 0 \Rightarrow \dot{P}^i \equiv F^i \tag{5.50}$$

thus introducing the concept of 'force'. You now see that the second law of Newton is nothing but a reflection of the existence or non-existence of a certain symmetry in Nature.

Newton's third law is also related to this idea: action-reaction pairs cancel each other so that the total force on an isolated system is zero and hence the total momentum is conserved. To see this, look back at the two particle system on a rail described by the Lagrangian (4.69). Using once again (5.32), we get

$$\delta S = 0 \Rightarrow P^i = m_1 \dot{q}_1 + m_2 \dot{q}_2 \; ; \tag{5.51}$$

Thus, *total* momentum is our Noether charge and it is conserved. The forces on each particle are $-\partial V/\partial q_1$ and $-\partial V/\partial q_2$, which are equal in magnitude but opposite in sign since we have $V(q_1 - q_2)$ — note the relative minus sign between q_1 and q_2

$$\frac{\partial V(q_1 - q_2)}{\partial q_1} = -\frac{\partial V(q_1 - q_2)}{\partial q_2} \; . \tag{5.52}$$

These two forces form the action-reaction pair. The cancelation of forces arises because of the dependence of the potential and force on the distance $q_1 - q_2$ between the particles — which is what makes the problem translationally invariant as well. We now see that the third law is intimately tied to the statement of translation symmetry.

EXAMPLE 5-6: Time translation and energy

Next, let us consider time translational invariance. Due to its particular usefulness, we want to treat this example with greater generality. We focus on a system with an arbitrary number of degrees of freedom labeled by q_k's with a general Lagrangian $L(q, \dot{q}, t)$. We propose the transformation

$$\delta t = \epsilon \quad , \quad \delta q_k = 0 \ . \tag{5.53}$$

Hence, the degrees of freedom are left unchanged, but the time is shifted by a constant ϵ. This means that we need a direct shift

$$\delta q_k = 0 = \Delta q_k + \dot{q}_k \delta t = \Delta q_k + \epsilon\, \dot{q}_k \Rightarrow \Delta q_k = -\epsilon\, \dot{q}_k \tag{5.54}$$

to compensate for the indirect change in q_k induced by the shift in time. We then use (5.32) to find the condition for time translational symmetry

$$\delta S = \int dt \left(-\epsilon \dot{q}_k \frac{\partial L}{\partial q_k} - \epsilon \ddot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \epsilon \frac{dL}{dt} \right) \ . \tag{5.55}$$

But we also know

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \frac{\partial L}{\partial q_k} \dot{q}_k + \frac{\partial L}{\partial \dot{q}_k} \ddot{q}_k \ . \tag{5.56}$$

We then get

$$\delta S = \int dt \, \frac{\partial L}{\partial t} \ . \tag{5.57}$$

In general, since L depends on the q_k's as well, we need to consider the more restrictive condition for symmetry $\delta S = 0$, *i.e.* we have $K =$ constant. This implies that we have time translational symmetry if

$$\frac{\partial L}{\partial t} = 0 \tag{5.58}$$

i.e. if the Lagrangian does not depend on time *explicitly*. If this is the case, we then have a conserved quantity given by (5.38)

$$Q = -\epsilon\, \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} + \epsilon\, L \ . \tag{5.59}$$

Dropping an overall constant term $-\epsilon$ and rearranging, we write

$$Q \to H = \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \ . \tag{5.60}$$

H is called the **Hamiltonian** of the system. Consider the two particle problem in one dimension described by the Lagrangian (4.69). One then finds

$$H = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}m_2\dot{q}_2^2 + V(q_1 - q_2) = E\ ; \tag{5.61}$$

otherwise known as **energy**. Hence, if we note that the results of an experiment do not change with the time schedule of the observation, we would expect that there is a quantity — called energy by definition — that remains constant in time. We can then look at dissipative effects that involve loss of energy and learn about new physics through the non-conservation of energy.

EXAMPLE 5-7: Rotations and angular momentum

Consider the problem of a non-relativistic particle of mass m moving in two dimensions, on a plane labeled by $x^1 = x$ and $y^1 = y$. We add to the problem a central force and write a Lagrangian of the form

$$L = \frac{1}{2}m\dot{x}^i\dot{x}^i - V(x^i x^i)\ . \tag{5.62}$$

Note that the potential depends only on the distance of the particle from the origin of the coordinate system $\sqrt{x^i x^i}$. Rotations are described by (5.21). We can then use (5.32) to test for rotational symmetry

$$\begin{aligned}\delta S &= \int dt\left(\frac{\partial V}{\partial x^i}\Delta x^i + m\,\dot{x}^i\frac{d}{dt}(\Delta x^i)\right)\\ &= \int dt\,\left(2\,V'\theta\,\varepsilon^{ij}x^i x^j + m\,\theta\,\varepsilon^{ij}\dot{x}^i\dot{x}^j\right)\ .\end{aligned} \tag{5.63}$$

In the second line, we wrote

$$\frac{\partial V}{\partial x^i} = \frac{\partial V}{\partial u}\frac{\partial u}{\partial x^i} = \frac{\partial V}{\partial u}(2\,x^i) = 2V'x^i \tag{5.64}$$

where $u \equiv x^i x^i$ and we used the chain rule. We now want to show that $\delta S = 0$. Focus on the first term in (5.63)

$$2\,V'\theta\,\varepsilon^{ij}x^i x^j = 2\,V'\theta\,\varepsilon^{12}x^1x^2 + 2\,V'\theta\,\varepsilon^{21}x^2x^1 = 2\,V'\theta\,x^1x^2 - 2\,V'\theta\,x^2x^1 = 0\ . \tag{5.65}$$

Let us do this one more time, with more grace and elegance:

$$\begin{aligned}&2\,V'\theta\,\varepsilon^{ij}x^i x^j\\ &= 2\,V'\theta\,\varepsilon^{ji}x^j x^i\\ &= 2\,V'\theta\,\varepsilon^{ji}x^i x^j\\ &= -2\,V'\theta\,\varepsilon^{ij}x^i x^j\ .\end{aligned} \tag{5.66}$$

In the second line, we just relabeled the indices $i \to j$ and $j \to i$: since they are summed indices, it does not matter what they are called. In the third line, we use the fact that multiplication is commutative $x^j x^i = x^i x^j$. Finally, in the third line, we used the property $\varepsilon^{ij} = -\varepsilon^{ji}$ from (5.22). Hence, we have shown

$$2\,V'\theta\,\varepsilon^{ij}x^i x^j = -2\,V'\theta\,\varepsilon^{ij}x^i x^j \Rightarrow 2\,V'\theta\,\varepsilon^{ij}x^i x^j = 0\ . \tag{5.67}$$

The key idea is that ε^{ij} is antisymmetric in its indices while $x^i x^j$ is symmetric under the same indices. The sum of their product then cancels. The same is true for the second term in (5.63)

$$m\,\theta\,\varepsilon^{ij}\dot{x}^i\dot{x}^j = -m\,\theta\,\varepsilon^{ij}\dot{x}^i\dot{x}^j \Rightarrow m\,\theta\,\varepsilon^{ij}\dot{x}^i\dot{x}^j = 0\ . \tag{5.68}$$

We will use this trick occasionally later on in other contexts. We thus have shown that our system is rotational symmetric

$$\delta S = 0 \Rightarrow K = \text{Constant}\ . \tag{5.69}$$

We can then determine the conserved quantity using (5.38)

$$Q = \frac{\partial L}{\partial \dot{x}^i}\Delta x^i = m\,\dot{x}^i\theta\,\varepsilon^{ij}x^j\ . \tag{5.70}$$

Dropping a constant term θ, we write

$$l = m\,\varepsilon^{ij}\dot{x}^i x^j = m\,\left(\dot{x}^1 x^2 - \dot{x}^2 x^1\right) = (\vec{\mathbf{r}} \times m\,\vec{\mathbf{v}})_z \tag{5.71}$$

i.e. this is the z-component of the angular momentum of the particle; it points perpendicular to the plane of motion as expected. Rotational symmetry implies conservation of angular momentum. Rotation about the z axis corresponds to angular momentum along the z axis.

EXAMPLE 5-8: Lorentz and Galilean Boosts

How about a Lorentz transformation? The Relativity postulate *requires* the Lorentz transformation as a symmetry of any physical system: it is not a question of whether it is a symmetry of a given system; it better be! We could then use (5.32), with Lorentz transformations, as a *condition* for sensible Lagrangians. Noether's theorem can be used to construct theories consistent with the required symmetries. In general, an experiment would identify a set of symmetries in a newly discovered system. Then the theorist's task is to build a Lagrangian that describes the system; and a good starting point would be to assure that the Lagrangian has all the needed symmetries. You now see the power of Noether's theorem: it allows you to mold equations and theories to your needs.

Coming back to Lorentz transformations, let us look at an explicit example and find the associated conserved charge. We would consider a relativistic system, say a free relativistic particle with action

$$S = -mc^2 \int d\tau = -mc^2 \int \sqrt{1 - \frac{\dot{x}^i \dot{x}^i}{c^2}} \ . \tag{5.72}$$

A particular Lorentz transformation is given by (5.26) which we can then substitute in (5.32) to show that we have a symmetry. However, this is an algebraically cumbersome exercise that we would prefer to leave as a homework problem for the reader. It is more instructive to simplify the problem further. Hence, we consider instead the small speed limit, *i.e.* we consider a non-relativistic system with Galilean symmetry. We take a single free particle in one dimension with Lagrangian

$$L = \frac{1}{2} m \dot{x}^2 \ . \tag{5.73}$$

The expected symmetry is Galilean, given by (1.1) which we quote again for convenience

$$x = x' + V t' \quad , \quad y = y' \quad , \quad z = z' \quad , \quad t = t' \ . \tag{5.74}$$

We can then write the infinitesimal version quickly

$$\begin{aligned} &\delta t = \delta y = \delta z = 0 \quad , \quad \delta x = -V\,t \\ &\Rightarrow \Delta t = \Delta y = \Delta z = 0 \text{ and } \Delta x = -V\,t \ . \end{aligned} \tag{5.75}$$

Note that we want to think of this transformation near $t \sim 0$, the instant in time the two origins coincide, to keep it as a small deformation for any V. Using (5.32), we then get

$$\delta S = \int dt\, m\, V\, \dot{x} \ . \tag{5.76}$$

Before we panic from the fact that this did not vanish, let us remember that all that is needed is $\delta S = \int dt\, dK/dt$ for some K. That is indeed the case

$$\delta S = \int dt \frac{d}{dt} \left(m\, V\, x \right) \ . \tag{5.77}$$

We then have

$$K = m\, V\, x + \text{Constant} \ . \tag{5.78}$$

The system then has Galilean symmetry. We look at the associated Noether charge using (5.38)

$$Q = \frac{\partial L}{\partial \dot{x}} \Delta x = m\,\dot{x}\,V\,t\ . \tag{5.79}$$

But this is not the conserved charge: $Q - K$ is the conserved quantity

$$Q - K = m\,\dot{x}\,V\,t - m\,V\,x = \text{Constant}\ . \tag{5.80}$$

Rewriting things, we have a simple first order differential equation

$$\dot{x}\,t - x = \text{Constant}\ . \tag{5.81}$$

Integrating this gives the expected linear trajectory $x(t) \propto t$. Unlike momentum, energy, and angular momentum, this conserved quantity $Q - K$ is not given its own glorified name. Since Galilean (or the more general Lorentz) symmetry is expected to be prevalent in all systems, this does not add any useful distinguishing physics ingredient to a problem. Perhaps if we were to discover a fundamental phenomenon that breaks Galilean/Lorentz symmetry, we could then revisit this conserved quantity and study its non-conservation. For now, this conserved charge gets relegated to second rate status...

EXAMPLE 5-9: Sculpting Lagrangians from symmetry

Let us take the previous example one step further. What if we were to *require* a symmetry and ask for all possible Lagrangians that fit the mold. To be more specific, consider a one dimensional system with one degree of freedom, denoted as $q(t)$. We want to ask: what are all possible theories that can describe this system with the following conditions: they are to be Galilean invariant and invariant under time translations. The Galilean symmetry is obtained from (5.75) in the previous example

$$\delta t = 0 \text{ and } \delta x = \Delta x = -V\,t\ . \tag{5.82}$$

Substituting this into (5.32), we get

$$\delta S = \int dt\,\left(-\frac{\partial L}{\partial x} V\,t - \frac{\partial L}{\partial \dot{x}} V\right) = \int dt\,\frac{d}{dt}\left(K\right)\ . \tag{5.83}$$

The question is then to find the most general L that does the job for some K. This means we need

$$\frac{\partial L}{\partial x} t + \frac{\partial L}{\partial \dot{x}} \propto \frac{d}{dt}\left(K\right) \to \frac{d}{dt}\left(\widetilde{K}\right)\ , \tag{5.84}$$

where we absorbed the proportionality constant inside the yet to be determined function $\widetilde{K}$. Note also that we are *not* allowed to use the equations of motion! Using the chain rule with $\widetilde{K}(t, x, \dot{x}, \ddot{x}, \ldots)$, we can write

$$\frac{d}{dt}\left(\widetilde{K}\right) = \frac{\partial \widetilde{K}}{\partial t} + \frac{\partial \widetilde{K}}{\partial x}\dot{x} + \frac{\partial \widetilde{K}}{\partial \dot{x}}\ddot{x} + \ldots \tag{5.85}$$

Comparing this to (5.84), we see that we need $\widetilde{K}(t, x)$ — a function of t and x only — since we know L is a function of t, x, and $\dot{x}$ only; hence we have

$$\frac{\partial L}{\partial x}t + \frac{\partial L}{\partial \dot{x}} = \frac{\partial \widetilde{K}}{\partial t} + \frac{\partial \widetilde{K}}{\partial x}\dot{x} \ . \tag{5.86}$$

We want a general form for L, yet $\widetilde{K}(t, x)$ is also arbitrary. Since the right-hand side is linear in $\dot{x}$, so must be the left hand side. This implies we need L to be a quadratic polynomial in $\dot{x}$

$$L = f_1(t, x)\dot{x}^2 + f_2(t, x)\dot{x} + f_3(t, x) \tag{5.87}$$

with three unknown functions $f_1(t, x)$, $f_2(t, x)$, and $f_3(t, x)$. Looking at the $\partial L/\partial x$ term, we can immediately see that we need $f_1(t, x) = C_1$, a constant independent of x and t: otherwise we generate a term quadratic in $\dot{x}$ that does not exist on the right-hand side of (5.86). Our Lagrangian now looks like

$$L = C_1\dot{x}^2 + f_2(t, x)\dot{x} + f_3(t, x) \ . \tag{5.88}$$

But we forgot about the time translational symmetry. That, we know, requires that $\partial L/\partial t = 0$. We then should write instead

$$L = C_1\dot{x}^2 + f_2(x)\dot{x} + f_3(x) \ . \tag{5.89}$$

But the second term is irrelevant to the dynamics. This is because L will appear in the action integrated over time; and this second term can be integrated out

$$\int dt\, f_2(x)\dot{x} = \int dt\, \frac{d}{dt}\left(F_2(x)\right) = F_2(x)|_{boundaries} \tag{5.90}$$

for some function $F_2(x) = \int^x d\xi\, f_2(\xi)$. Hence the term does not depend on the shape of paths plugged into the action functional and cannot contribute to the statement of extremization — otherwise known as the equation of motion. We are now left with

$$L \to C_1\dot{x}^2 + f_3(x) \ . \tag{5.91}$$

The condition (5.86) on L now looks like

$$\frac{\partial f_3(x)}{\partial x} t + 2\, C_1 \dot{x} = \frac{\partial \widetilde{K}}{\partial t} + \frac{\partial \widetilde{K}}{\partial x} \dot{x} \ . \tag{5.92}$$

Picking out the $\dot{x}$ dependences on either sides, this implies

$$2\, C_1 = \frac{\partial \widetilde{K}}{\partial x} \quad , \quad \frac{\partial f_3(x)}{\partial x} t = \frac{\partial \widetilde{K}}{\partial t} \ . \tag{5.93}$$

Since we know that

$$\frac{\partial^2 \widetilde{K}}{\partial x \partial t} = \frac{\partial^2 \widetilde{K}}{\partial t \partial x} \ , \tag{5.94}$$

differentiating the two equations in (5.93) leads to the condition

$$\frac{\partial^2 f_3(x)}{\partial x^2} = 0 \Rightarrow f_3(x) = C_2 x + C_3 \tag{5.95}$$

for some constants C_2 and C_3. The Lagrangian now looks like

$$L = C_1 \dot{x}^2 + C_2 x \ , \tag{5.96}$$

where we set $C_3 = 0$ since a constant shift of L does not affect the equation of motion. We can now solve for $\widetilde{K}$ as well if we wanted to using (5.93)

$$\widetilde{K} = 2\, C_1 x + \frac{C_2}{2} t^2 + \text{Constant} \ . \tag{5.97}$$

Now, let us stare back at the important point, equation (5.96). We write the constants C_1 and C_2 as $C_1 = m/2$ and $C_2 = -m\, g$

$$L = \frac{1}{2} m \dot{x}^2 - m\, g\, x \ . \tag{5.98}$$

With tears of joy in our eyes, we just showed that the most general Galilean and time translation invariant mechanics problem in one dimension necessarily looks like a particle in uniform gravity. We were able to *derive* the canonical kinetic energy term and gravitational potential from a symmetry requirement. This is just a hint at the power of symmetries and conservation laws in physics. Indeed, all the known forces of Nature can be derived from first principles from symmetries!

EXAMPLE 5-10: Generalized momenta and cyclic coordinates

For a generalized coordinates q_k, we define the **generalized momentum** p_k to be

$$p_k \equiv \frac{\partial L}{\partial \dot{q}_k}. \tag{5.99}$$

In terms of p_k, the Lagrange equations are

$$\frac{dp_k}{dt} = \frac{\partial L}{\partial q_k}. \tag{5.100}$$

Let us see why we call the p_k's **momenta**. If a particular coordinate q_k is *absent* from the Lagrangian, we know that the transformation $\delta q_k =$ constant will be a symmetry of our system — as can be seen explicitly using (5.32). The associated conserved Noether charge is given by (5.38) and is nothing but p_k. The equations of motion (5.100) are telling us this already since $\partial L/\partial q_k = 0$. Hence we are justified to call the p_k's generalized momenta. Such a coordinate q_k missing from a Lagrangian is said to be a **cyclic** coordinate or an *ignorable* coordinate. For any cyclic coordinate the Lagrange equations (5.100) tell us that the time derivative of the corresponding generalized momentum is zero, so that particular generalized momentum is *conserved*.

Take for example the problem of a particle free to move on a tabletop under the influence of a central spring force, as discussed in an earlier example. Using polar coordinates, the Lagrangian in that case was

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - \frac{1}{2}kr^2, \tag{5.101}$$

which contains both $\dot{r}$ and $\dot{\varphi}$, and also the coordinate r, but *not* the coordinate φ. In the problem of the particle on the end of the spring, the generalized momentum is the angular momentum of the particle, which is indeed conserved

$$p^\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\, r^2\, \dot{\varphi}\ . \tag{5.102}$$

One of the first things to notice about a Lagrangian is whether there are any cyclic coordinates, because any such coordinate leads to a conservation law that is also a first integral of motion. This means that the equation of motion for that coordinate is already half solved, in that it is a first-order differential equation rather than the second-order differential equation one typically gets for a noncyclic coordinate.

5.2.2 Some comments on symmetries

Let us step back for a moment and comment on several additional issues about symmetries and conservation laws.

- If a system has N degrees of freedom, then the typical Lagrangian leads to N second order differential equations (provided the Lagrangian depends on at most first derivatives of the variables). If we were lucky enough to solve these equations, we would parameterize the solution with $2N$ constants related to the boundary conditions. If our system has M symmetries, it would then have M conserved quantities. Each of the symmetries leads to a first order differential equation, and hence a total of M constants of motion. In total, the conservation equations will gives has $2M$ constants to parameterize the solution with: M from the constants of motion, and another M for integrating the first order equations. These $2M$ constants would necessarily be related to the $2N$ constants mentioned earlier. What if we have $M = N$? The system is then said to be **integrable**. This means that all one needs to do is to write the conservation equations and integrate them. We need not even stare at any second order differential equations to find the solution to the dynamics. In general, we will have $M \leq N$. And the closer M is to N, the easier will be to understand the given physical problem. As soon as a good physicist sees a physics problem, he or she would first count degrees of freedom; then he or she would instinctively look for the symmetries and associated conserved charges. This immediately lays out a strategy how to tackle the problem based on how much symmetries one has versus the number of degrees of freedom.

- Noether's theorem is based on infinitesimal transformations: symmetries that can be build up from small incremental steps of deformations. There are other symmetries in Nature that do not fit this prescription. For example, discrete symmetries are rather common. Reflection transformations, *e.g. time reflection* $t \to -t$ *or discrete rotations of a lattice*, can be very important for understanding the physics of a problem. Noether's theorem does not apply to these. However, such symmetries are also often associated with conserved quantities. Sometimes, these are called **topological** conservation laws.

- Infinitesimal transformations can be catalogued rigorously in mathematics. A large and useful class of such transformations fall under the topic of **Lie groups** of Group theory. The Lie group catalogue (developed by Cartan) is exhaustive. Many if not all of the entries in the catalogue are indeed realized in Nature in various physical systems. In addition to Lie groups, physicists also flirt with other more exotic symmetries such as supersymmetry. Albeit mathematically very beautiful, unfortunately none of these will be of interest to us in mechanics.

5.3 Exercises and Problems

PROBLEM 5-1 :

Consider a particle of mass m moving in two dimensions in the $x-y$ plane, constrained to a rail-track whose shape is describe by an arbitrary function $y = f(x)$. There is NO GRAVITY acting on the particle.

(a) Write the Lagrangian in terms of the x degree of freedom only.

(b) Consider some general transformation of the form

$$\delta x = g(x) \quad , \quad \delta t = 0 \; ; \tag{5.103}$$

where $g(x)$ is an arbitrary function of x. Assuming that this transformation is a symmetry of the system such that $\delta S = 0$, show that this implies the following differential equation relating $f(x)$ and $g(x)$

$$\frac{g'}{g} = -\frac{1}{2(1+f'^2)}\frac{d}{dx}\left(f'^2\right) \; ; \tag{5.104}$$

where prime stands for derivative with respect to x (not t).

(c) Write a general expression for the associated conserved charge in terms of $f(x)$, $g(x)$, and $\dot{x}$.

(d) We will now specify a certain $g(x)$, and try to find the laws of physics obeying the prescribed symmetry; *i.e.* for given $g(x)$, we want to find the shape of the rail-track $f(x)$. Let

$$g(x) = \frac{g_0}{\sqrt{x}} \; ; \tag{5.105}$$

where g_0 is a constant. Find the corresponding $f(x)$ such that this $g(x)$ yields a symmetry. Sketch the shape of the rail-track. (HINT: $h(x) = f'^2$.)

PROBLEM 5-2 : One of the most important symmetries in Nature is that of **Scale Invariance**. This symmetry is very common (e.g. arises whenever a substance undergoes phase transition), fundamental (e.g. it is at the foundation of the concept of Renormalization Group for which a physics Nobel Prize was awarded in 1982), and entertaining (as you will now see in this problem).

Consider the action

$$S = \int dt \, \sqrt{h}\dot{q}^2 \tag{5.106}$$

of two degrees of freedom $h(t)$ and $q(t)$.

(a) Show that the following transformation (known as a scale transformation or dilatation)

$$\delta q = \alpha q \quad , \quad \delta h = -2\alpha h \quad , \quad \delta t = \alpha t \tag{5.107}$$

is a symmetry of this system.

(b) Find the resulting constant of motion.

PROBLEM 5-3 :

A massive particle moves under the acceleration of gravity and without friction on the surface of an inverted cone of revolution with half angle α.

(a) Find the Lagrangian in polar coordinates.

(b) Provide a complete analysis of the trajectory problem. Use Noether charges when useful.