Classical Mechanics
for the 19th century

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Chapter 1

Newtonian particle mechanics

In this chapter we review the laws of Newtonian mechanics. We set the stage with inertial frames and the Galilean transformation, move on to Newton's celebrated three laws of motion, present a catalogue of forces that are commonly encountered in mechanics, and end with a review of dimensional analysis. All this is a preview to a relativistic treatment of mechanics in the next chapter.

1.1 Inertial frames and the Galilean transformation

Classical mechanics begins by analyzing the motion of *particles*. Classical particles are idealizations: they are pointlike, with no internal degrees of freedom like vibrations or rotations. But by understanding the motion of these ideal "particles" we can also understand a lot about the motion of *real* objects, because we can often ignore what is going on inside of them. The concept of "classical particle" can in the right circumstances be used for objects all the way from electrons to baseballs to stars to entire galaxies.

In describing the motion of a particle we first have to choose a frame of reference in which an observer can make measurements. Many reference frames could be used, but there is a special set of frames, the nonaccelerating, **inertial frames**, which are particularly simple. Picture a set of three orthogonal meter sticks defining a set of Cartesian coordinates drifting through space with no forces applied. The set of meter sticks neither accelerates nor rotates *relative to visible distant stars.* An inertial observer drifts with the

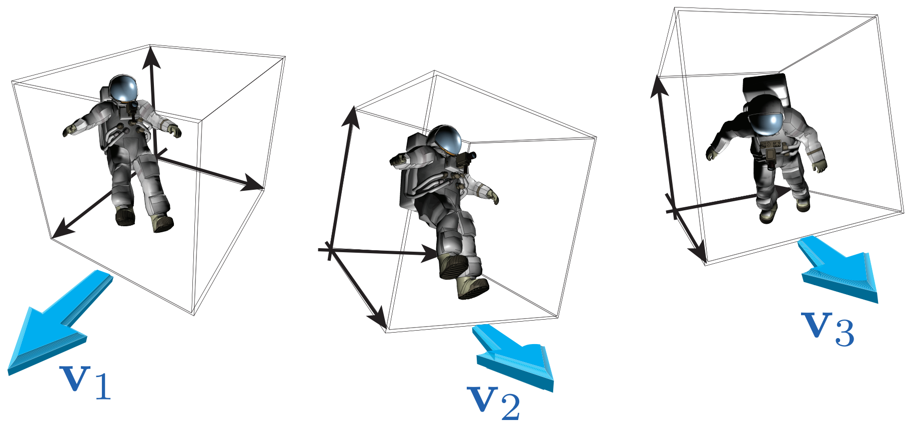


Figure 1.1: Various inertial frames in space. If one of these frames is inertial, any other frame moving at constant velocity relative to it is also inertial.

coordinate system and uses it to make measurements of physical phenomena. This inertial frame and inertial observer are not unique, however: having established one inertial frame, any other frame moving at constant velocity relative to it is also inertial, as illustrated in Figure 1.1.

Two of these inertial observers, along with their personal coordinate systems, are depicted in Figure 1.2: observer $\mathcal{O}$ describes positions of objects through a Cartesian system labeled by (x, y, z), while observer $\mathcal{O}'$ uses a system labeled by (x', y', z').

An **event** of interest to an observer is characterized by the position in space at which the measurement is made — but also by the instant in time at which the observation occurs, according to clocks at rest in the observer's inertial frame. For example, an event could be a snapshot in time of the position of a particle along its trajectory. Hence, the event is assigned four numbers by observer $\mathcal{O}$: x, y, z, and t for time, while observer $\mathcal{O}'$ labels the same event x', y', z', and t'.

Without loss of generality, observer $\mathcal{O}$ can choose her x axis along the direction of motion of $\mathcal{O}'$, and then the x' axis of $\mathcal{O}'$ can be aligned with that axis as well, as shown in Figure 1.2. it seems intuitively obvious that the two coordinate systems are then related by

$$x = x' + Vt' \ , \quad y = y' \ , \quad z = z' \quad t = t' \tag{1.1}$$

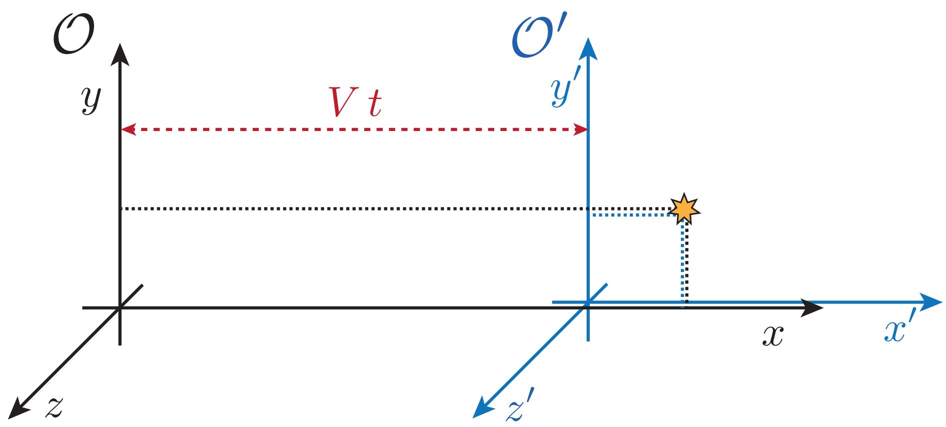


Figure 1.2: Two inertial frames, $\mathcal{O}$ and $\mathcal{O}'$, moving relative to one another along their mutual x axes.

where we assume that the origins of the two frames coincide at time $t' = t = 0$. This is known as a **Galilean transformation**. Note that the only difference in the coordinates is in the x direction, corresponding to the distance between the two origins as each system moves relative to the other. This transformation — in spite of being highly intuitive — will turn out to be incorrect, as we shall see in the next chapter. But for now, we take it as good enough for our Newtonian purposes.

If the coordinates represent the instantaneous position of a particle, we can write

$$x(t) = x'(t') + Vt' \ , \quad y(t) = y'(t') \ , \quad z(t) = z'(t') \quad t = t' \ . \tag{1.2}$$

We then differentiate this transformation with respect to $t = t'$ to obtain the transformation laws of velocity and acceleration. Differentiating once gives

$$v^x = v'^x + V \ , \quad v^y = v'^y \ , \quad v^z = v'^z \ , \tag{1.3}$$

where for example $v^x \equiv dx/dt$ and $v'_x \equiv dx'/dt'$, and differentiating a second time gives

$$a^x = a'^x \ , \quad a^y = a'^y \ , \quad a^z = a'^z . \tag{1.4}$$

That is, the velocity components of a particle differ by the relative frame velocity in each direction, while the acceleration components are the same in every inertial frame. Therefore one says that the acceleration of a particle is **Galilean invariant**.

We take as a postulate that the fundamental laws of classical mechanics are also Galilean invariant. This mathematical statement is equivalent to the physical statement that an observer at rest in any inertial frame is qualified to use the fundamental laws – there is no preferred inertial frame of reference. This equivalence of inertial frames is called the **principle of relativity**.

We are now equipped to summarize the fundamental laws of Newtonian mechanics, and discuss their invariance under the Galilean transformation.

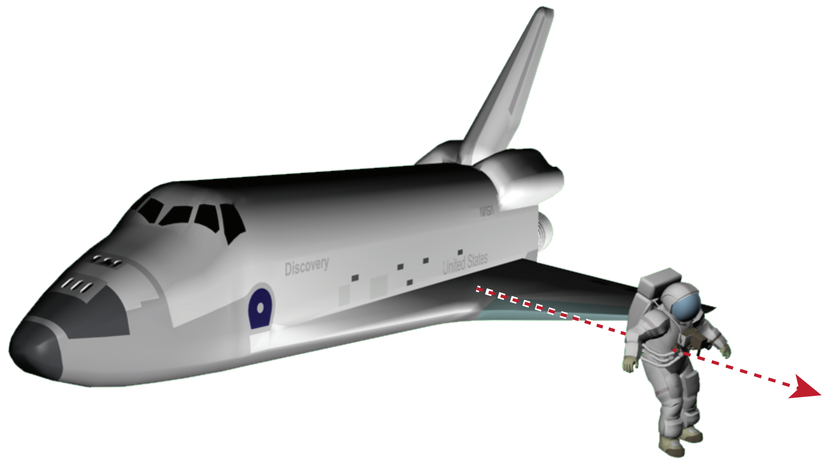
1.2 Newton's laws of motion

In his *Principia* of 1687, Isaac Newton (1642-1727) presented his famous three laws. The first of these is the **law of inertia**:

I: If there are no forces on an object, then if the object starts at rest it will stay at rest, or if it is initially set in motion, it will continue moving in the same direction in a straight line at constant speed.

We can use this definition to test whether or not our frame is inertial. If we are inertial observers and we remove all interactions from a particle under observation, if set at rest the particle will stay put, and if tossed in any direction it will keep moving in that direction with constant speed. The law of inertia is obeyed, so by definition our frame is inertial. Note from the Galilean velocity transformation that if a particle has constant velocity in one inertial frame it has constant velocity in all inertial frames. *Hence, Galilean transformations correctly connect the perspectives of inertial reference frames.*

An astronaut set adrift from her spacecraft in outer space, far from Earth, or the Sun, or any other gravitating object, will move off in a straight line at constant speed when viewed from an inertial frame. So if her spaceship is drifting without power and is not rotating, the spaceship frame is inertial, and onboard observers will see her move away in a straight line. But if her spaceship is rotating, for example, observers on the ship will see her move off in a curved path — the frame inside a rotating spaceship is not inertial.



Now consider an inertial observer who observes a particle to which a force **F** is applied. Then **Newton's second law states that**

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{1.5}$$

where the momentum of the particle is $\mathbf{p} = m\mathbf{v}$**, the product of its mass and velocity. That is,**

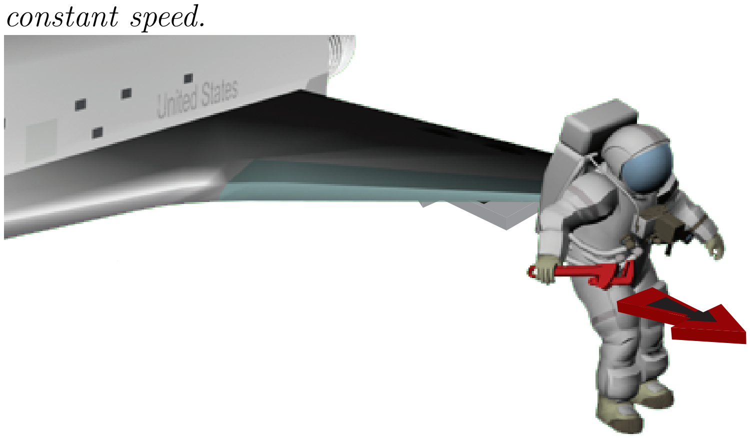
II: The time rate of change of a particle's momentum is equal to the net force on that particle.

Newton's second law tells us that if the momentum of a particle changes, there must be a net force causing that change. Note that the second law gives us the means to identify and quantify the effect of forces and interactions. By conducting a series of measurements of the rate of change of momenta of a selection of particles, we explore the forces acting on them in their environment. Once we understand the nature of these forces, we can use this knowledge to predict the motion of other particles in a wider range of circumstances — this time by deducing the effect of such forces on rate of change of momentum.

Note that the derivative $d\mathbf{p}/dt = md\mathbf{v}/dt = m\mathbf{a}$, so Newton's second law can also be written in the form $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a}$ is the acceleration of the particle. The law therefore implies that if we remove all forces from an object, neither its momentum nor its velocity will change: it will remain at rest if started at rest, and move in a straight line at constant speed if given an initial velocity. But that is just Newton's *first* law, so it might seem that the first law is just a special case of the second law! However, the second law is not true in all frames of reference. An accelerating observer will see the momentum of an object changing, even if there is no net force on it. In fact, it is only inertial observers who can use Newton's second law, so the first law is not so much a special case of the second as a means of specifying those observers for whom the second law is valid.

Newton's second law is the most famous fundamental law of classical mechanics, and it must also be Galilean invariant according to our principle of relativity. We have already shown that the acceleration of a particle is invariant and we also take the mass of a particle to be the same in all inertial frames. So if $\mathbf{F} = m\mathbf{a}$ is to be a fundamental law, which can be used by observers at rest in any inertial frame, we must insist that the force on a particle is likewise a Galilean invariant. Newton's second law itself does not specify which forces exist, but in classical mechanics any force on a particle (due to a spring, gravity, friction, or whatever) must be the same in all inertial frames.

If the drifting astronaut is carrying a wrench, by throwing it away (say) in the forward direction she exerts a force on it. During the throw the momentum of the wrench changes, and after it is released, it travels in some straight line at constant speed.

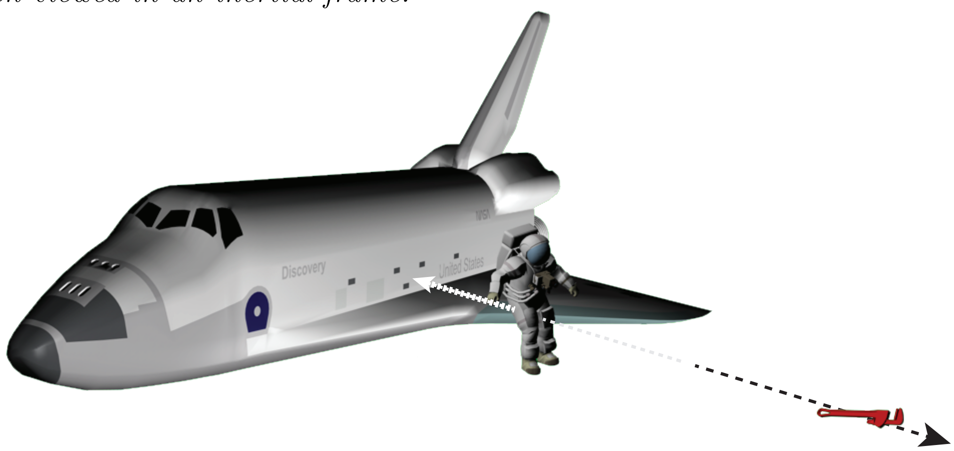


Finally, Newton's *third* law states that

III: "Action equals reaction". If one particle exerts a force on a second particle, the second particle exerts an equal but opposite force back on the first particle.

We have already stated that any force acting on a particle in classical mechanics must be the same in all inertial frames, so it follows that Newton's third law is also Galilean invariant: a pair of equal and opposite forces in a given inertial frame transform to the same equal and opposite pair in another inertial frame.

While the astronaut, drifting away from her spaceship, is exerting a force on the wrench, at each instant the wrench is exerting an equal but opposite force back on the astronaut. This causes the astronaut's momentum to change as well, and if the change is large enough her momentum will be reversed, allowing her to drift back to her spacecraft in a straight line at constant speed when viewed in an inertial frame.



EXAMPLE 1-1: A bacterium with a viscous retarding force

The most important force on a nonswimming bacterium in a fluid is the viscous drag force $F = -bv$, where v is the velocity of the bacterium relative to the fluid and b is a constant that depends on the size and shape of the bacterium and the viscosity of the fluid — the minus sign means that the drag force is opposite to the direction of motion. If the bacterium, as illustrated in Figure 1.3, gains a velocity v_0 and then stops swimming, what is its subsequent velocity as a function of time?

Let us assume that the fluid defines an inertial reference frame. Newton's second law then leads to the ordinary differential equation

$$m\frac{dv}{dt} = -b\,v \Rightarrow m\,\ddot{x} = -b\,\dot{x} \tag{1.6}$$

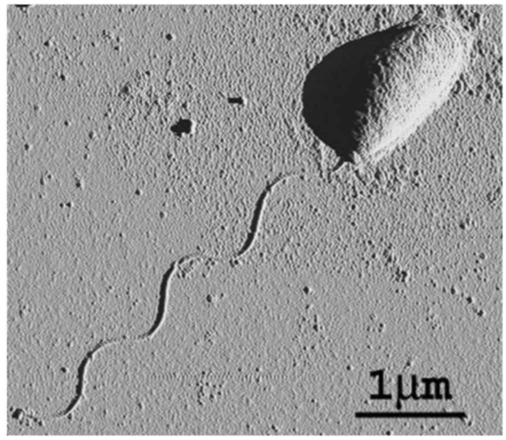


Figure 1.3: A bacterium in a fluid. What is its motion if it begins with velocity v_0 and then stops swimming?

where $\dot{x} \equiv dx/dt$ and $\ddot{x} \equiv d^2x/dt^2$. As is always the case with Newton's second law, this is a second-order differential equation in position. However, it is a particularly simple one that can be integrated at once. Separating variables and integrating,

$$\int_{v_0}^{v} \frac{dv}{v} = -\frac{b}{m} \int_0^t dt, \tag{1.7}$$

which gives $\ln(v) - \ln(v_0) = \ln(v/v_0) = -(b/m)t$. Exponentiating both sides,

$$v = v_0 e^{-(b/m)t} \equiv v_0 e^{-t/\tau} \tag{1.8}$$

where $\tau \equiv m/b$ is called the "time constant" of the exponential decay. In a single time constant, *i.e.*, when $t = \tau$, the velocity decreases to $1/e$ of its initial value; therefore τ is a measure of how quickly the bacterium slows down. The bigger the drag force (or the smaller the mass) the greater the deceleration.

An alternate way to solve the differential equation is to note that it is linear with constant coefficients, so the exponential form $v(t) = Ae^{\alpha t}$ is bound to work, for an *arbitrary* constant A and a *particular* constant α. In fact, the constant $\alpha = -1/\tau$, found by substituting $v(t) = Ae^{\alpha t}$ into the differential equation and requiring that it be a solution. In this first-order differential equation, the constant A is the single required arbitrary constant. It can be determined by imposing the initial condition $v = v_0$ at $t = 0$, which tells us that $A = v_0$.

Now we can integrate once again to find the bacterium's position $x(t)$. If we choose the x direction to be in the $\mathbf{v}_0$ direction, then $v = dx/dt$, so

$$x(t) = v_0 \int_0^t e^{-t/\tau}\ dt = v_0 \tau \left(1 - e^{-t/\tau}\right). \tag{1.9}$$

The bacterium's starting position is $x(0) = 0$, and as $t \to \infty$, its position x asymptotically approaches the value $v_0\tau$. Note that given a starting position and an initial velocity, the path of a bacterium is determined by the drag force exerted on it and the condition that it stop swimming.

EXAMPLE 1-2: A linearly damped oscillator

We seek to find the motion of a mass m confined to move in the x direction at one end of a Hooke's-law spring of force-constant k, and which is also subject to the damping force $-b\,v$ where b is a constant. That is, we assume that the damping force is linearly proportional to the velocity of the mass and in the direction opposite to its motion. This is seldom true for macroscopic objects: damping is usually a steeper function of velocity than this. However, linear (i.e., viscous) damping illustrates the general features of damping while permitting us to find exact analytic solutions.

Newton's second law gives

$$F = -kx - b\dot{x} = m\ddot{x}, \tag{1.10}$$

a second-order linear differential equation equivalent to

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \tag{1.11}$$

where we write $\beta \equiv b/2m$ and $\omega_0 \equiv \sqrt{k/m}$ to simplify the notation. We see here a scenario that is typical in a problem using Newton's second law: we generate a second-order differential equation. Mathematically, we are guaranteed a solution once we fix two *initial conditions*. This can be, for example, the initial position $x(0) = x_0$ and the initial velocity $v(t) = \dot{x}(t) = v_0$. Hence, our solution will depend on two constants to be specified by the particular problem. In general, each dynamical variable we track through

Newton's second law will generate a single second-order differential equation, and hence will require two initial conditions. This is the sense in which Newton's laws provide us with predictive power: fix a few constants using initial conditions, and physics will tell us the future evolution of the system. For the example at hand, equation (1.11) is a *linear* differential equation with constant coefficients, which can be solved by setting $x \propto e^{\alpha t}$ for some α. Substituting this form into equation (1.11) gives the quadratic equation

$$\alpha^2 + 2\beta\alpha + \omega_0^2 = 0 \tag{1.12}$$

with solutions

$$\alpha = -\beta \pm \sqrt{\beta^2 - \omega_0^2}. \tag{1.13}$$

There are now three possibilities: (1) $\beta > \omega_0$, the "overdamped" solution; (2) $\beta < \omega_0$, the "underdamped" solution; and (3) $\beta = \omega_0$, the "critically damped" solution.

(1) In the overdamped case the exponent α is *real* and *negative*, and so the position of the mass as a function of time is

$$x(t) = A_1 e^{-\gamma_1 t} + A_2 e^{-\gamma_2 t} \tag{1.14}$$

where $\gamma_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$ and $\gamma_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$. Here A_1 and A_2 are arbitrary constants. The two terms are the expected linearly independent solutions of the second-order differential equation, and the coefficients A_1 and A_2 can be determined from the initial position x_0 and initial velocity v_0 of the mass. Figure 1.4(a) shows a plot of $x(t)$.

(2) In the underdamped case, the quantity $\sqrt{\beta^2 - \omega_0^2} = i\sqrt{\omega_0^2 - \beta^2}$ is purely *imaginary*, so

$$x(t) = e^{-\beta t}(A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t}) \tag{1.15}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. We can use **Euler's identity**

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{1.16}$$

to write x in terms of purely real functions,

$$x(t) = e^{-\beta t}(\bar{A}_1 \cos\omega_1 t + \bar{A}_2 \sin\omega_1 t) \tag{1.17}$$

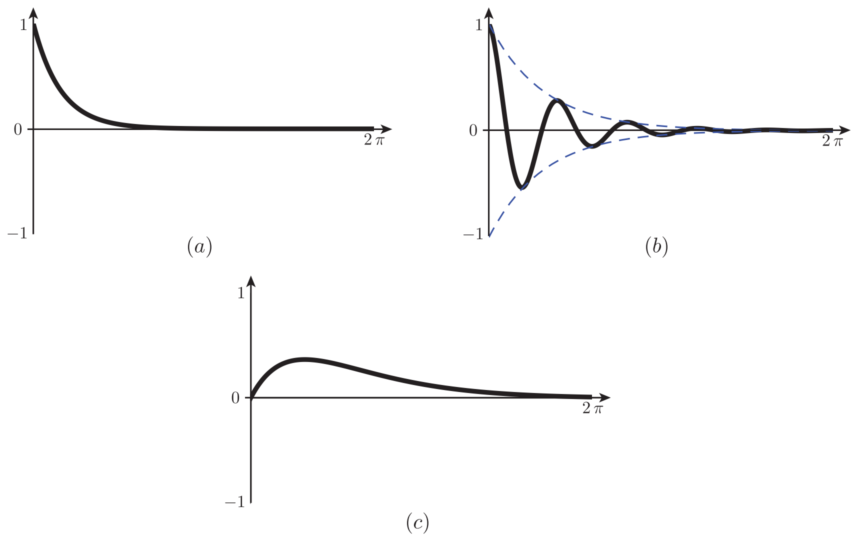


Figure 1.4: Motion of an oscillator if it is (a) overdamped, (b) underdamped, or (c) critically damped, for the special case where the oscillator is released from rest ($v_0 = 0$) at some position x_0.

where $\bar{A}_1 = A_1 + A_2$ and $\bar{A}_2 = i(A_1 - A_2)$ are real coefficients. We can also use the identity $\cos(\theta + \varphi) = \cos\theta\cos\varphi - \sin\theta\sin\varphi$ to write equation (1.17) in the form

$$x(t) = A\,e^{-\beta t}\cos(\omega_1 t + \varphi) \tag{1.18}$$

where $A = \sqrt{\bar{A}_1^2 + \bar{A}_2^2}$ and $\varphi = \tan^{-1}(-\bar{A}_2/\bar{A}_1)$. That is, the underdamped solution corresponds to a decaying oscillation with amplitude $A\,e^{-\beta t}$. The arbitrary constants A and φ can be determined from the initial position x_0 and velocity v_0 of the mass. Figure 1.4(b) shows a plot of $x(t)$. If there is no damping at all, we have $b = \beta = 0$ (and the oscillator is obviously "underdamped"). The original equation (1.11) becomes the *simple harmonic oscillator equation*

$$\ddot{x} + \omega_0^2 x = 0 \tag{1.19}$$

whose most general solution is

$$x(t) = A\cos(\omega_0 t + \varphi). \tag{1.20}$$

This gives away the meaning of ω_0: it is the angular frequency of oscillation of a simple harmonic oscillator, related to the oscillation frequency ν in cycles/second by $\omega_0 = 2\pi\nu$. Note that $\omega_1 < \omega_0$; *i.e.* the damping slows down the oscillations in addition to damping the amplitude.

(3) In the critically damped case $\beta = \omega_0$ the two solutions of equation (1.12) merge into the single solution $x(t) = A\,e^{-\beta t}$. A second-order differential equation has two linearly independent solutions, however, so we need one more. This additional solution is $x = A' t\, e^{-\beta t}$ for an arbitrary coefficient A', as can be seen by substituting this form into equation (1.11). The general solution for the critically damped case is therefore

$$x = (A + A't)e^{-\beta t} \tag{1.21}$$

which has the two independent constants A and A' needed to provide a solution determined by the initial position x_0 and velocity v_0. Figure 1.4(c) shows a plot of $x(t)$ in this case.

Whichever solution applies, it is clear that the motion of the particle is determined by (a) the initial position $x(0)$ and velocity $\dot{x}(0)$, and (b) the force acting on it throughout its motion.

1.3 Systems of particles

So far we have concentrated on *single particles*. We will now expand our horizons to encompass systems of an arbitrary number of particles. A system of particles might be an entire solid object like a bowling ball, in which tiny parts of the ball can be viewed as individual infinitesimal particles. Or we might have a liquid in a glass, or the air in a room, or a planetary system, or a galaxy of stars, all made of constituents we treat as 'particles'.

The location of the i^{th} particle of a system can be identified by a position vector $\mathbf{r}_i$ extending from the origin of coordinates to that particle, as illustrated in Figure 1.5. Using the laws of classical mechanics for each particle in the system, we can find the laws that govern the system as a whole. Define the total momentum $\mathbf{P}$ of the system to be the sum of the momenta of the individual particles,

$$\mathbf{P} = \sum_i \mathbf{p}_i. \tag{1.22}$$

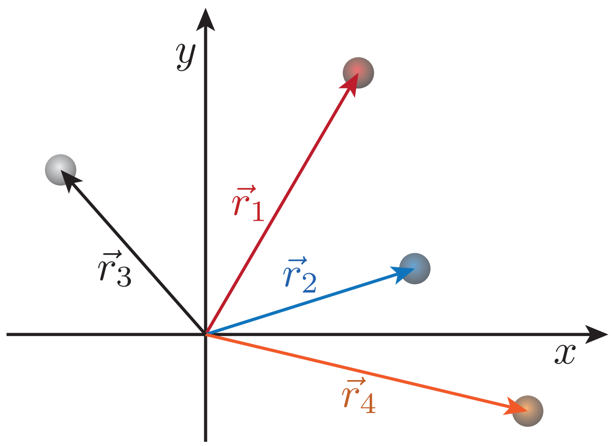


Figure 1.5: A system of particles, with each particle identified by a position vector **r**

Similarly, define the total force $\mathbf{F}_T$ on the system to be the sum of all the forces on all the particles,

$$\mathbf{F}_T = \sum_i \mathbf{F}_i. \tag{1.23}$$

It then follows that $\mathbf{F}_T = d\mathbf{P}/dt$, just by adding up the individual $\mathbf{F}_i = d\mathbf{p}_i/dt$ equations for all the particles. If we further split up the total force $\mathbf{F}_T$ into $\mathbf{F}_{\text{ext}}$ (the sum of the forces exerted by external agents, like Earth's gravity or air resistance on the system of particles that form a golfball) and $\mathbf{F}_{\text{int}}$ (the sum of the internal forces between members of the system themselves, like the mutual forces between particles within the golfball), then

$$\mathbf{F}_T = \mathbf{F}_{\text{int}} + \mathbf{F}_{\text{ext}} = \mathbf{F}_{\text{ext}}, \tag{1.24}$$

because all the internal forces cancel out by Newton's third law. That is, for any two particles i and j, the force of i on j is equal but opposite to the force of j on i. Finally, we can write a grand second law for the system as a whole,

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}}{dt} \tag{1.25}$$

showing how the system as a whole moves in response to external forces.

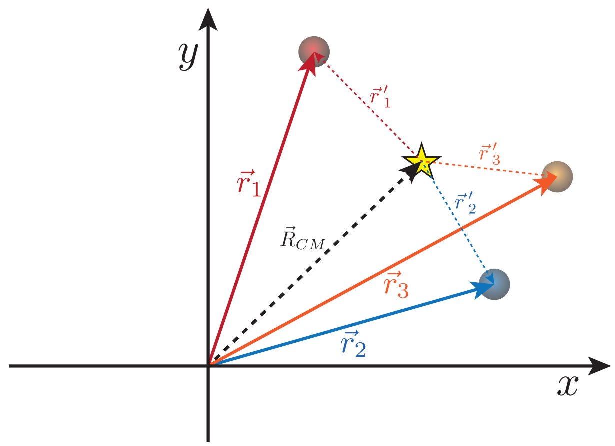


Figure 1.6: A collection of particles, each with a position vector $\mathbf{r}_i$ from a fixed origin. The center of mass $\mathbf{R}_{CM}$ is shown, and also the position vector $\mathbf{r}'_i$ of the ith particle measured from the center of mass.

Now the importance of momentum is clear. For if no external forces act on the collection of particles, their *total* momentum cannot depend upon time, so $\mathbf{P}$ is conserved. Individual particles in the collection may move in complicated ways, but they always move in such a way as to keep the total momentum constant.

A useful quantity characterizing a system of particles is their center of mass position $\mathbf{R}_{CM}$. Let the i^{th} particle have mass m_i, and define the center of mass of the collection of particles to be

$$\mathbf{R}_{CM} = \frac{\sum_i m_i \mathbf{r}_i}{M}, \tag{1.26}$$

where $M = \sum_i m_i$ is the total mass of the system. We can write the position vector of a particle as the sum $\mathbf{r}_i = \mathbf{R}_{CM} + \mathbf{r}'_i$, where $\mathbf{r}'_i$ is the position vector of the particle measured from the center of mass, as illustrated in Figure 1.6.

The velocity of the center of mass is

$$\mathbf{V}_{CM} = \frac{d\mathbf{R}_{CM}}{dt} = \frac{\sum_i m_i \mathbf{v}_i}{M} = \frac{\mathbf{P}}{M} \tag{1.27}$$

differentiating term by term, and using the fact that the particle masses are constant. Again $\mathbf{P}$ is the total momentum of the particles, so we have proven that the center of mass moves at constant velocity whenever $\mathbf{P}$ is conserved–that is, whenever there is no net external force. In particular, if there is no external force on the particles, their center of mass stays at rest if it starts at rest.

This result is also very important because it shows that a real object composed of many smaller "particles" can be considered a particle itself: it obeys all of Newton's laws with a position vector given by $\mathbf{R}_{CM}$*, a momentum given by* $\mathbf{P}$*, and the only relevant forces being the external ones. It relieves us of having to draw a distinct line between* particles *and* systems *of particles. For some purposes we think of a star as composed of many smaller particles, and for other purposes the star as a whole could be considered to be a single particle in the system of stars called a galaxy.*

1.4 Conservation laws

Using Newton's laws we can show that under the right circumstances, there are as many as three dynamical properties of a particle that remain constant in time, *i.e.* that are *conserved*. These properties are **momentum, angular momentum,** and **energy**. They are conserved under different circumstances, so in any particular case all of them, none of them, or only one or two of them may apply. As we will see, a conservation law typically leads to a *first-order* differential equation, which is generally much easier to tackle that the usual second-order equations we get from Newton's second law. This makes identifying conservation laws in a system a powerful tool for problem solving and characterizing the motion. We will later also learn in Chapter 5 that there are deep connections between conservation laws and symmetries in Nature.

MOMENTUM

From Newton's second law it follows that if there is no net force on a particle, its momentum $\mathbf{p} = m\mathbf{v}$ is conserved, so its velocity $\mathbf{v}$ is also constant. Conservation of momentum for a single particle simply means that a free particle (a particle with no force on it) moves in a straight line at constant speed. For a single particle, conservation of momentum is equivalent to Newton's first law.

For a *system* of particles, however, momentum conservation becomes non-trivial, because it requires only the conservation of *total* momentum $\mathbf{P}$. When there are no *external* forces acting on a system of particles, the total momentum of the individual constituents remains constant, even though the momentum of each single particle may change:

$$\mathbf{P} = \sum_i \mathbf{p}_i = \text{constant}. \tag{1.28}$$

As we saw earlier, this is the momentum of the center of mass of the system if we were to imagine the sum of all the constituent masses added up and placed at the center of mass. This relation can be very handy when dealing with several particles.

EXAMPLE 1-3: A wrench in space

We are sitting within a spaceship watching a colleague astronaut outside holding a wrench. The astronaut-plus-wrench system is initially at rest from our point of view. The angry astronaut (of mass M) suddenly throws the wrench (of mass m), with some unknown force. We then see the astronaut moving with velocity $\mathbf{V}$. Without knowing anything about the force with which she threw the wrench, we can compute the velocity of the wrench. No external forces act on the system consisting of wrench plus astronaut, so its total momentum is conserved:

$$\mathbf{P} = M\,\mathbf{V} + m\,\mathbf{v} = \text{constant}, \tag{1.29}$$

where $\mathbf{v}$ is the unknown velocity of the wrench. Since the system was initially at rest, we know that $\mathbf{P} = 0$ for all time. We then deduce

$$\mathbf{v} = -\frac{M\,\mathbf{V}}{m} \tag{1.30}$$

without needing to even stare at Newton's second law or any other second-order differential equation.

ANGULAR MOMENTUM

Let a position vector $\mathbf{r}$ extend from an origin of coordinates to a particle, as shown in Figure 1.7. The angular momentum of the particle is defined to be

$$\boldsymbol{\ell} = \mathbf{r} \times \mathbf{p}, \tag{1.31}$$

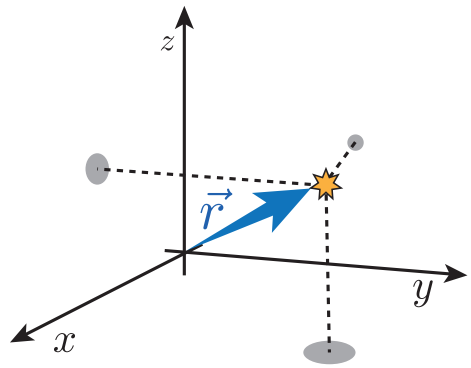


Figure 1.7: The position vector for a particle. Angular momentum is always defined with respect to a chosen point from where the position vector originates.

the vector cross product of $\mathbf{r}$ with the particle's momentum $\mathbf{p}$. Note that in a given inertial frame the angular momentum of the particle depends not only on properties of the particle itself, namely its mass and velocity, *but also upon our choice of origin*. Using the product rule, the time derivative of $\boldsymbol{\ell}$ is

$$\frac{d\boldsymbol{\ell}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt}. \tag{1.32}$$

The first term on the right is $\mathbf{v} \times m\mathbf{v}$, which vanishes because the cross product of two parallel vectors is zero. In the second term, we have $d\mathbf{p}/dt = \mathbf{F}$ using Newton's second law, where $\mathbf{F}$ is the net force acting on the particle. It is therefore convenient to define the **torque** $\mathbf{N}$ on the particle due to $\mathbf{F}$ as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}, \tag{1.33}$$

so that

$$\mathbf{N} = \frac{d\boldsymbol{\ell}}{dt}. \tag{1.34}$$

That is, the net *torque* on a particle is responsible for any change in its angular momentum, just as the net *force* on the particle is responsible for any change in its momentum. The angular momentum of a particle is conserved if there is no net torque on it.

Sometimes the momentum $\mathbf{p}$ is called the "linear momentum" to distinguish it from the angular momentum $\boldsymbol{\ell}$. However, *linear momentum and angular momentum are not two aspects of the same thing.* They have different units and are conserved under different circumstances. The momentum of a particle is conserved if there is no net external *force* and the angular momentum of the particle is conserved if there is no net external *torque.* It is easy to arrange forces on an object so that it experiences a net force but no net torque, and equally easy to arrange them so there is a net torque but no net force. For example, if the force $\mathbf{F}$ is parallel to $\mathbf{r}$, we have $\mathbf{N} = 0$; yet there is a non-zero force.

There is another striking difference between momentum and angular momentum: in a given inertial frame, the value of a particle's momentum $\mathbf{p}$ is independent of where we choose to place the origin of coordinates. But because the angular momentum $\boldsymbol{\ell}$ of the particle involves the position vector $\mathbf{r}$, the value of $\boldsymbol{\ell}$ does depend on the choice of origin. This makes angular momentum somewhat more abstract than momentum, in that in the exact same problem different people at rest in the same inertial frame may assign it different values depending on where they choose to place the origin of their coordinate system.

EXAMPLE 1-4: A particle moving in two dimensions with an attractive spring force

A block of mass m is free to move on a frictionless tabletop under the influence of an attractive Hooke's-law spring force $\mathbf{F} = -k\mathbf{r}$, where the vector $\mathbf{r}$ is the position vector of the particle measured from the origin. We will find the motion $x(t), y(t)$ of the ball and show that the angular momentum of the ball about the origin is conserved.

The vector $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$, where x and y are the Cartesian coordinates of the ball and $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors pointing in the positive x and positive y directions, respectively. Newton's second law $-k\mathbf{r} = m\ddot{\mathbf{r}}$ becomes

$$-k(x\hat{\mathbf{x}} + y\hat{\mathbf{y}}) = m(\ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}}), \tag{1.35}$$

which separates into the two simple harmonic oscillator equations

$$\ddot{x} + \omega_0^2 x = 0 \qquad \text{and} \qquad \ddot{y} + \omega_0^2 y = 0 \tag{1.36}$$

where $\omega_0 = \sqrt{k/m}$. It is interesting that the x and y motions are completely independent of one another in this case; the two coordinates have been decoupled, so we can solve the equations separately. The solutions are

$$x = A_1 \cos(\omega_0 t + \varphi_1) \qquad \text{and} \qquad y = A_2 \cos(\omega_0 t + \varphi_2), \tag{1.37}$$

showing that the ball oscillates simple harmonically in both directions. The four constants $A_1, A_2, \varphi_1, \varphi_2$ can be evaluated in terms of the four initial conditions x_0, y_0, v^{x0}, v^{y0}. The oscillation frequencies are the same in each direction, so orbits of the ball are all closed. In fact, the orbit shapes are ellipses centered at the origin, as shown in Figure 1.8[1]. Note that in this two-dimensional problem, the motion of the ball is determined by *four* initial conditions (the two components of the position vector and the two components of the velocity vector) together with the known force throughout the motion. This is what is expected for two second-order differential equations.

The spring exerts a torque on the ball about the origin, which is $\mathbf{N} = \mathbf{r}\times\mathbf{F} = \mathbf{r}\times -k\mathbf{r} = 0$, since the cross product of any vector with itself vanishes. Therefore the angular momentum of the ball is conserved about the origin. In this case, this angular momentum is given by

$$\boldsymbol{\ell} = (x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}) \times (m\,\dot{x}\,\hat{\mathbf{x}} + m\,\dot{y}\,\hat{\mathbf{y}}) = (m\,x\,\dot{y} - m\,y\,\dot{x})\hat{\mathbf{z}}\ , \tag{1.39}$$

so the special combination $m\,x\,\dot{y} - m\,y\,\dot{x}$ remains constant for all time. That is certainly a highly non-trivial statement.

The angular momentum is *not* conserved about any other point in the plane, because then the position vector and the force vector would be neither parallel nor antiparallel. The angular momentum of a particle is *always* conserved if the force is purely central, *i.e.* if it is always directly toward or away from a fixed point, as long as that same point is chosen as origin of the coordinate system.

[1] Remember that the equation of an ellipse in the $x - y$ plane can be written as

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{1.38}$$

where (x_0, y_0) is the center of the ellipse, and a and b are the minor and major radii. One can show that equation (1.37) indeed satisfies this equation for appropriate relations between φ_1, φ_2, A_1, A_2 and x_0, y_0, a, b.

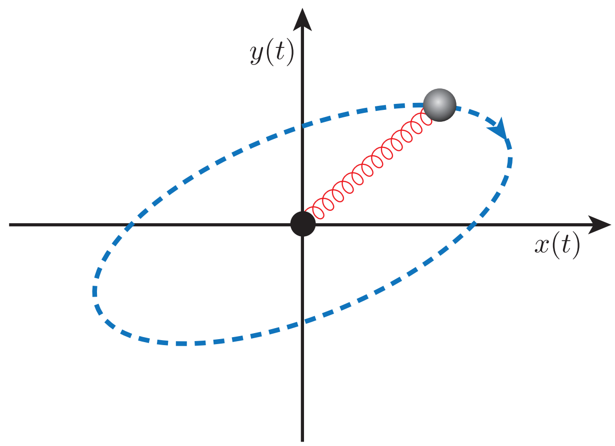


Figure 1.8: A two-dimensional elliptical orbit of a ball subject to a Hooke's law spring force, with one end of the spring fixed at the origin.

We still have not used the conservation of angular momentum in this problem to our advantage, because we solved the full second-order differential equation. To see how we can tackle this problem without ever needing to stare at Newton's second law or any second-order differential equation, we need to first look at another very useful conservation law, energy conservation.

ENERGY

The **energy** of a particle is a third quantity that is sometimes conserved. Let $\mathbf{F}$ be one of the forces acting on a particle. Define the **work** done by $\mathbf{F}$ on the particle to be the line integral

$$W_{ab} = \int_a^b \mathbf{F} \cdot d\mathbf{s} \tag{1.40}$$

as the particle moves between two points a and b by whatever path it happens to take. Here $d\mathbf{s}$ is a vector that points in the direction of the path at some point and whose magnitude ds is an infinitesimal distance along the path at that point. The dot product $\mathbf{F} \cdot d\mathbf{s} = F ds \cos\theta$, where θ is the angle between the vectors $\mathbf{F}$ and $d\mathbf{s}$, which shows that it is only the component of $\mathbf{F}$ *parallel* to the path at some point that does work on the particle. Figure 1.9 illustrates the setup.

If we were to integrate Newton's second law along the path of a particle acted upon by a single force, we would write

$$W_{ab} = \int_a^b \mathbf{F} \cdot d\mathbf{s} = \int_a^b m\mathbf{a} \cdot d\mathbf{s} \ . \tag{1.41}$$

The right-hand side may be simplified further

$$\int_a^b m\mathbf{a} \cdot d\mathbf{s} = \int_a^b m\frac{d\mathbf{v}}{dt} \cdot d\mathbf{s} = \int_a^b md\mathbf{v} \cdot \frac{d\mathbf{s}}{dt} = \int_a^b md\mathbf{v} \cdot \mathbf{v} \ . \tag{1.42}$$

In the last step, we moved the dt to $d\mathbf{s}$ to change the measure of the integral to $d\mathbf{v}$. We can now easily integrate the right-hand side

$$\int_a^b md\mathbf{v} \cdot d\mathbf{v} = \frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 \ , \tag{1.43}$$

as can be easily verified by expanding the dot product $d\mathbf{v} \cdot \mathbf{v} = dv^x v^x + dv^y v^y + dv^z v^z$. Defining **Kinetic Energy** as

$$T \equiv \frac{1}{2}mv^2 \ , \tag{1.44}$$

we now have an "integral form" of Newton's second law

$$W_{ab} = T_b - T_a \ . \tag{1.45}$$

That is, the work done by the force is the difference between the final and initial kinetic energies of the particle.

Often the work done by a particular force $\mathbf{F}$ depends upon which path the particle takes as it moves from a to b. The frictional work done by air resistance on a ball as it flies from the bat to an outfielder depends upon how high it goes, that is, whether its total path length is short or long. There are other forces, however, like the static force of gravity, for which the work done is independent of the particle's path: For example, the work done by Earth's gravity on the ball is the same no matter how it gets to the outfielder. For such forces the work depends only upon the endpoints a and b. That implies that the work can be written as the difference

$$W_{ab} = -U(b) + U(a) \tag{1.46}$$

between a "potential energy" function U evaluated at the final point b and the initial point a. Similarly, the work done by this force as the particle

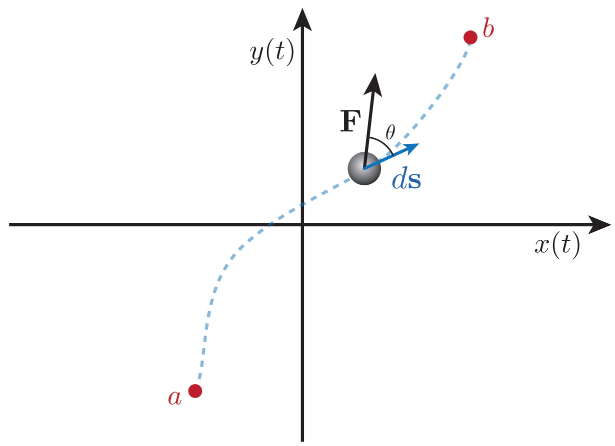


Figure 1.9: The work done by a force on a particle is its line integral along the path traced by the particle.

moves from b to a third point c is $W_{bc} = -U(c) + U(b)$, so the work done as the particle moves all the way from a to c is

$$W_{ac} = W_{ab} + W_{bc} = -U(b) + U(a) - U(c) + U(b) = -U(c) + U(a) \quad (1.47)$$

as expected, independent of the intermediate point b.

Forces $\mathbf{F}$ for which the work $W_{ab} = \int_a^b \mathbf{F} \cdot d\mathbf{s}$ between any two points a and b is independent of the path, are said to be **conservative**. There are several tests for conservative forces that are mathematically equivalent, in that if any one of them is true the others are true as well. The conditions are

(1) $W_{ab} = \int_a^b \mathbf{F} \cdot d\mathbf{s}$ is path independent.

(2) The integral around any closed path $\oint \mathbf{F} \cdot d\mathbf{s} = 0$.

(3) The curl of the force function vanishes: $\nabla \times \mathbf{F} = 0$.

(4) The force function can always be written as the negative gradient of some scalar function U: $\mathbf{F} = -\nabla U$.

Often the third of these conditions makes the easiest test. For example, the curl of the uniform gravitational force $\mathbf{F} = -mg\,\hat{\mathbf{z}}$ is, using the determinant expression for the curl,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F^x & F^y & F^z \end{vmatrix} = 0, \tag{1.48}$$

since each component of $\mathbf{F}$ is zero or a constant. Therefore this force is conservative. On the other hand, the curl of the hypothetical force $\mathbf{F} = \alpha xy\hat{\mathbf{z}}$, where α is a constant, is

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & \alpha xy \end{vmatrix} \\ &= \hat{\mathbf{x}}\frac{\partial}{\partial y}(\alpha\, x\, y) - \hat{\mathbf{y}}\frac{\partial}{\partial x}(\alpha\, x\, y) = \alpha(x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}) \neq 0, \end{aligned} \tag{1.49}$$

so this force is not conservative, and does not possess a potential energy function.

The work done is equal to the difference between two potential energies, so it follows that the physics is exactly the same for a particle with potential energy $U(\mathbf{r})$ as it is for a potential energy $U(\mathbf{r})+C$, where C is any constant. For example, the potential energy of a particle of mass m in a uniform gravitational field g is $U_{\text{grav}} = mgh$, where h is the altitude of the particle. The fact that any constant can be added to U in this case is equivalent to the fact that it doesn't matter from what point the altitude is measured; the motion of a particle is the same whether we measure altitude from the ground or from the top of a building.

Putting all this together, for a system involving conservative forces, we can now write

$$W_{ab} = -U(b) - U(a) = T_b - T_a\ . \tag{1.50}$$

Introducing a new quantity we call **energy**, we then write

$$E \equiv T + U \Rightarrow E_b = E_a\ , \tag{1.51}$$

that is energy as the sum of kinetic and potential energies is conserved in systems involving only conservative forces. Otherwise, we could write a more general statement

$$W_{ab}^{\text{noncons}} = E_b - E_a\ ; \tag{1.52}$$

i.e. the work done by non-conservative forces measures the non-conservation of energy in the system. It has turned out to be very useful to expand the concept of energy beyond kinetic and potential energies, so in the case of nonconservative forces like friction, a decrease in the "mechanical energy" $T + U$ shows up in some other form, such as heat. That is, conservation of energy is more general than one might expect from classical mechanics alone; in addition to kinetic and potential energies, there is thermal energy, the energy of deformation, energy in the electromagnetic field, and many other forms as well. Energy is often a useful concept across many disparate physical systems. We will later on see a more appropriate and physical definition of the notion energy. For now however, the statement of conservation of $T + U$ will be very useful.

It is instructive to reverse this exercise by taking the time derivative of E,

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\,\mathbf{v}\cdot\mathbf{v}\right) + \frac{dU}{dt} = m\,\mathbf{v}\cdot\mathbf{a} + \mathbf{v}\cdot\nabla\mathbf{U}\ . \tag{1.53}$$

In the first term we used the product rule on $\mathbf{v}\cdot\mathbf{v}$, and in the second term we used the chain rule

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \sum_i \frac{\partial U}{\partial x_i}\frac{dx^i}{dt} = \mathbf{v}\cdot\nabla U, \tag{1.54}$$

where we further assumed in the last step that U is a function of the coordinates only, $U(\mathbf{r})$, and not upon time or velocity. We now have

$$\frac{dE}{dt} = \mathbf{v}\cdot(m\,\mathbf{a}) + \mathbf{v}\cdot\nabla U = \mathbf{v}\cdot\mathbf{F} + \mathbf{v}\cdot(-\mathbf{F}) = 0\ , \tag{1.55}$$

where we used $\mathbf{F} = m\,\mathbf{a}$ and $\mathbf{F} = -\nabla U$, assuming that all the forces on the particle arise from the potential U and hence are conservative. We thus have come back full circle to Newton's second law. This process has made it clear however that the potential must only depend on the coordinates of the particle $U(\mathbf{r})$. Later on however, we will be able to extend this notion to certain forces involving velocity dependence such as the magnetic force.

EXAMPLE 1-5: A particle attached to a spring revisited

We want to demonstrate the power of conservation laws in the previous problem of a particle of mass m confined to a two-dimensional plane and attached to a spring of spring constant k (see Figure 1.8). The only force law is Hooke's law $\mathbf{F} = -k\mathbf{r}$. We can check that $\nabla \times \mathbf{F} = 0$, and then find that the potential energy for this conservative force is

$$U(b) - U(a) = -\int_a^b \mathbf{F} \cdot d\mathbf{r} = -k \int_a^b \mathbf{r} \cdot d\mathbf{r} \Rightarrow U = \frac{1}{2} k\, r^2 \ . \tag{1.56}$$

The total energy is therefore

$$E = \frac{1}{2} m\, v^2 + \frac{1}{2} k\, r^2 \ . \tag{1.57}$$

The problem has circular symmetry, so it is helpful to use polar coordinates. The velocity of the particle is

$$\mathbf{v} = \dot{r}\, \hat{\mathbf{r}} + r\, \dot{\theta}\, \hat{\boldsymbol{\theta}} \tag{1.58}$$

where r and θ are the polar coordinates (see Appendix A for a review of coordinate systems). We then have

$$E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + \frac{1}{2} k\, r^2 \ . \tag{1.59}$$

Since E is a constant, this would be a very nice *first-order* differential equation for $r(t)$ if we could get rid of the pesky $\dot{\theta}$ term. Angular momentum conservation comes to the rescue. We know

$$\mathbf{L} = \mathbf{r} \times (m\, \mathbf{v}) = m\, r\, \hat{\mathbf{r}} \times (\dot{r}\, \hat{\mathbf{r}} + r\, \dot{\theta}\, \hat{\boldsymbol{\theta}}) = m\, r^2\, \dot{\theta}\, \hat{\mathbf{z}} = \text{constant} \ . \tag{1.60}$$

We then can write

$$m\, r^2\, \dot{\theta} = L \Rightarrow \dot{\theta} = \frac{L}{m\, r^2} \tag{1.61}$$

with L a constant. Putting this back into (1.59), we get

$$E = \frac{1}{2} m\, \dot{r}^2 + \frac{L^2}{2\, m\, r^2} + \frac{1}{2} k\, r^2 \ , \tag{1.62}$$

which is a first-order differential equation that determines $r(t)$, from which we can find $\theta(t)$ using equation (1.61). We have thus solved the problem without ever dealing with a second-order differential equation arising from Newton's second law. In this case, this is not particularly advantageous, given that the original second-order differential equations corresponded to harmonic oscillators. In general, however, tackling first-order differential equations is likely to be much easier.

It is instructive to analyze the boundary conditions and conservation law of this system. Newtons' second law gives are two second order differential equations in two dimensions. Each differential equations requires two boundary conditions to yield a unique solution. That's a total of four constants to determine in tackling the problem through second order differential equations. Energy conservation on the other hand provides us with a single first order differential equation that requires one boundary condition. But the value of energy E is another constant to be specified. So, that is two constants to fix. Angular momentum conservation gives us another first order differential equations with one boundary condition, plus the value L for the angular momentum. Hence, that's another two constants. The energy and angular momentum conservation equations thus require a total of four constant to yield a unique solution, as expected from the perspective of the original second order differential equations. The four boundary conditions of Newton's second law are hence directly related to the four constants to fix in solving the problem through conservation equations.

EXAMPLE 1-6: Newtonian gravity and its potential energy

Newton's law of gravity for the force on a 'probe' particle of mass m due to a 'source' particle of mass M is $\mathbf{F} = -(GMm/r^2)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector pointing *from* the source particle *to* the probe (see Figure 1.4). The minus sign means that the force is attractive, in the negative $\hat{\mathbf{r}}$ direction. We can check to see whether this force is conservative by taking its curl,

$$\nabla \times \left(-\frac{GMm}{r^2}\hat{\mathbf{r}} \right) = \frac{1}{r\sin\theta}\frac{\partial F^r}{\partial\varphi} - \frac{1}{r}\frac{\partial F^r}{\partial\theta} = 0, \tag{1.63}$$

where we use spherical coordinates throughout (see Appendix A for a review of coordinate systems). So Newton's gravitational force is conservative, and must therefore have a corresponding potential energy function

$$U(r) = -\int \mathbf{F}\cdot d\mathbf{r} = GMm\int \frac{dr}{r^2} = -\frac{G\,M\,m}{r} + \text{constant}, \tag{1.64}$$

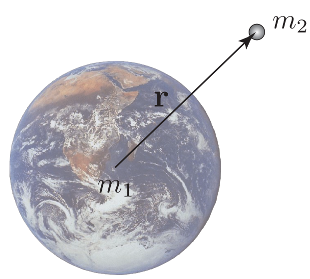


Figure 1.10: Newtonian gravity pulling a probe mass m towards a source mass M.

where by convention we ignore the constant of integration, which in effect sets the potential energy to be zero at infinity $r \to \infty$.

EXAMPLE 1-7: Dropping a particle in spherical gravity

Armed with the potential energy expression due to a spherical gravitating body, the total energy of a probe particle of mass m, which is conserved, is

$$E = T + U(r) = \frac{1}{2}mv^2 - \frac{G\,M\,m}{r}, \tag{1.65}$$

where M is the mass of the body. Suppose that the probe particle is dropped from rest some distance r_0 from the center of M, which we assume is so large that it does not move appreciably as the small mass m falls toward it. The particle has no initial tangential velocity, so it will fall radially with $v^2 = \dot{r}^2$. Energy conservation gives

$$E = \frac{1}{2}m\dot{r}^2 - \frac{G\,M\,m}{r}. \tag{1.66}$$

The initial conditions are $r = r_0$ and $\dot{r} = 0$, so it follows that $E = -GMm/r_0$.

Equation (1.66) is a first-order differential equation in $r(t)$. It is said to be a "first integral" of the second-order differential equation $\mathbf{F} = m\mathbf{a}$, which in this case is

$$-\frac{GMm}{r^2} = m\ddot{r}. \tag{1.67}$$

That is, if we want to find the motion $r(t)$ it is a great advantage to begin with energy conservation, because that equation already represents one integral of $\mathbf{F} = m\mathbf{a}$. Solving (1.66) for $\dot{r}$,

$$\dot{r} = \pm\sqrt{\frac{2}{m}\left(E + \frac{GMm}{r}\right)} = \pm\sqrt{2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}. \tag{1.68}$$

We have to choose the minus sign, because when the particle is released from rest it will subsequently fall toward the origin with $\dot{r} < 0$. Dividing the equation through by the right-hand side and integrating over time,

$$\int_{r_0}^{r} \frac{dr\sqrt{r}}{\sqrt{1 - r/r_0}} = -\sqrt{2\,G\,M}\int_0^t dt = -\sqrt{2\,G\,M}\;t. \tag{1.69}$$

At this point we say that the problem has been **reduced to quadrature**, an old-fashioned phrase that simply means that all that remains to find $r(t)$ (or in this case $t(r)$) is to evaluate an indefinite integral. If we are lucky, the integral can be evaluated in terms of known functions, in which case we have an *analytic* solution. If we are not so lucky, the integral can at least be evaluated numerically to any level of accuracy we need. An analytic solution to the integral in equation (1.69), using the substitution $r = r_0 \sin^2\theta$, gives

$$t(r) = \sqrt{\frac{r_0^3}{2\,G\,M}}\left[\frac{\pi}{2} - \sin^{-1}\sqrt{\frac{r}{r_0}} + \sqrt{\frac{r}{r_0}}\sqrt{1 - \frac{r}{r_0}}\right] \tag{1.70}$$

from which we can find the time it takes to fall to r given some initial value r_0. We cannot solve explicitly for $r(t)$ in this case, because the right-hand side is a transcendental function of r. Note that the constant r_0 in this equation is directly related to the energy E through equation (1.68).

The problem is much simplified if the particle falls from a great altitude to a much smaller altitude, so that $r \ll r_0$, in which case the first term in equation (1.70) is much bigger than the others. For example, the time it takes an astronaut to fall from rest at radius r_0 to the surface of an asteroid of radius R, where $r_0 \gg R$, is essentially

$$t = \frac{\pi}{2}\sqrt{\frac{r_0^3}{2GM}}, \tag{1.71}$$

which is independent of R! This insensitivity to the asteroid radius is due to the fact that nearly all of the travel time is spent at large radii, during which the astronaut is moving slowly. Changes in the asteroid radius R affect the overall travel time very little, because the astronaut is falling so fast near the end. On the other hand, the travel time is clearly quite sensitive to the initial position r_0.

EXAMPLE 1-8: Potential energies and turning points for positive power-law forces

A particle moves in one dimension subject to the power-law force $F = -kx^n$, where the coefficient k is positive, and n is a positive integer. Let us find the potential energy of the particle and also the maximum distance $x_{\max}$ it can reach from the origin, in terms of its maximum speed $v_{\max}$. The maximum distance is the "turning point" of the particle, because as the particle approaches this position it slows down, stops at $x_{\max}$, and turns around and heads in the opposite direction.

The potential energy of the particle is the indefinite integral

$$U = -\int^x F(x)dx = -\int^x (-kx^n)dx = \frac{k}{n+1}x^{n+1} \tag{1.72}$$

plus an arbitrary constant of integration, which we will choose to be zero. Two of these potential energy functions, one with odd n and one with even n, illustrate the range of possibilities, as shown in Figure 1.11. The case $n = 1$, corresponding to a linear restoring force, corresponds to a Hooke's-law spring, where k is the spring constant and the potential energy is $U = (1/2)kx^2$. In this case the lowest possible energy is $E = 0$, when the particle is stuck at $x = 0$. There are two turning points for energies $E > 0$, one at the right and one at the left.

The quadratic force with $n = 2$ has a cubic potential $U = (1/3)kx^3$ is positive for $x > 0$ and negative for $x < 0$, also as shown. Note that the slope

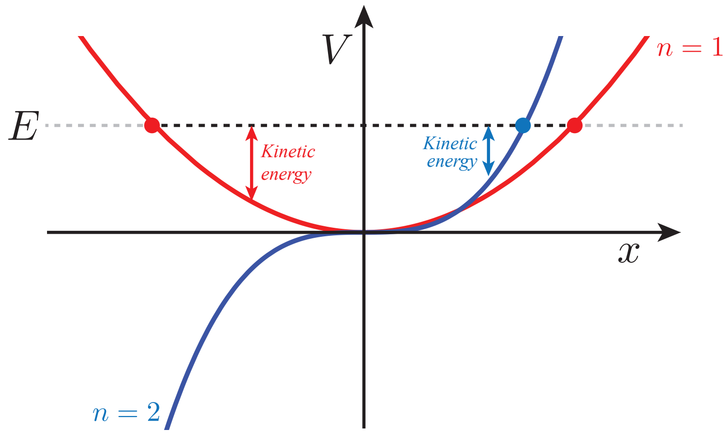


Figure 1.11: Potential energy functions for selected positive powers n. A possible energy E is drawn as a horizontal line, since E is constant. The difference between E and $U(x)$ at any point is the value of the kinetic energy T. The kinetic energy is zero at the "turning points", where the E line intersects $U(x)$. Note that for $n = 1$ there are two turning points for $E > 0$, but for $n = 2$ there is only a single turning point.

of this potential is everywhere positive except at $x = 0$, so the force on any particle at $x \neq 0$ is toward the left, since $F = -dU/dx$ is then negative. So particles at positive x are pulled toward the origin, while particles at negative x are pushed away from the origin in this case.

Energy is conserved for this entire set of forces, where

$$E = \frac{1}{2}mv^2 + \left(\frac{k}{n+1}\right)x^{n+1}. \tag{1.73}$$

The potential energy increases with increasing positive x, so the maximum speed of the particle is at the origin, where $E = (1/2)mv^2_{max}$. The speed goes to zero at the maximum value of x attainable, *i.e.*, where $E = k\,x^{n+1}_{max}/(n+1)$. Eliminating E and solving for x_{max},

$$x_{\text{max}} = \left(\frac{n+1}{2k}\right)^{1/(n+1)} (v_{\text{max}})^{2/(n+1)}. \tag{1.74}$$

For the spring force, with $n = 1$, x_{max} is directly proportional to v_{max}, so if we double the particle's velocity at the origin that will double the maximum x it can achieve.

Note that the conservation of energy equation (1.73) can be solved for $v \equiv \dot{x}$ to give

$$\dot{x} = \pm\sqrt{\frac{2}{m}\left(E - \left(\frac{k}{n+1}\right)x^{n+1}\right)}, \tag{1.75}$$

which is a first-order differential equation. Dividing by the right-hand side and integrating over time yields

$$\int\sqrt{\frac{dx}{E-(k/(n+1))x^{n+1}}} = \pm\sqrt{\frac{2}{m}}\int dt = \pm\sqrt{\frac{2}{m}}\,t + C, \tag{1.76}$$

where C is a constant of integration: The problem has been reduced to quadrature. For some values of n the integral on the left can be evaluated in terms of standard functions; this includes the cases $n = -1, 0$, and $+1$, for example. For other values of x the integral can be evaluated numerically. Note that conservation of energy results in a first-order differential equation, so specifying the constant of integration C is equivalent to specifying one initial condition.

Rather than integrating equation (1.73), which leads to equation (1.76), we can *differentiate* the equation instead. The time derivative of equation (1.73) is

$$m\dot{x}\ddot{x} + \left(\frac{k}{n+1}\right)(n+1)x^n\dot{x} = 0, \tag{1.77}$$

since $dE/dt = 0$. The velocity $\dot{x}$ is not generally zero, so we can divide it out, leaving

$$m\ddot{x} = -kx^n \tag{1.78}$$

which we recognize as $m\,a = F$ for the given force $F = -kx^n$. That is, the time derivative of the energy conservation first-order differential equation is simply $F = m\,a$, which is a second-order differential equation. Often energy conservation serves as a "first integral of motion", halfway toward a complete solution of the second-order equation $F = m\,a$.

1.5 Forces of nature

The hallmark of Newtonian mechanics — the relationship $\mathbf{F} = m\,\mathbf{a}$ — is only part of the physical content of a mechanics problem. To determine the dynamics of a particle, we still need the left-hand side of the equation: we need an independent specification of the forces. This is a separate physics statement that we need to discover and learn about through experimentation and additional theoretical considerations. We may then be tempted to ask the bold question: *what are all of the possible forces that can arise on the left-hand side of Newton's second law*? Surprisingly, this question has a complete answer, an exhaustive and finite catalogue of possibilities.

To date, we are aware of four, and only four, fundamental forces in nature — of which only two can be used in classical, Newtonian mechanics. For the sake of completeness, let us list all four:

1. The **electromagnetic force** can be attractive or repulsive, and acts only on particles that carry a certain mysterious attribute we call 'electric charge'. This force is relevant from subatomic length scales to planetary length scales, and it plays a role in virtually every physical setting.

2. The **gravitational force** is an omni-present, attractive force in classical physics, that acts on anything that has mass. Gravity is by far the weakest of the four forces, but at macroscopic length scales it is very noticeable nonetheless if objects are essentially electrically neutral, so that the much stronger electromagnetic force vanishes.

3. The **weak force** is subatomic in nature, acting only over very short distances (around 10^{-15} meters!), where it is essential to use quantum mechanics; the weak force therefore plays no role in typical classical mechanics problems. The weak force is important for understanding radioactivity, neutrinos, and the ever-elusive Higgs boson particle. We have also learned recently that the weak force is closely related to electromagnetism. The electromagnetic and weak forces collectively are sometimes referred to as the **electroweak** force.

4. The **strong force**, which is also a force of subatomic relevance (around 10^{-18} meters!), binding quarks together and underlying all nuclear energy. This is the strongest of all the forces, but in spite of its great importance it is not directly relevant to classical mechanics.

Gravity and electromagnetism are the two mainstays of classical mechanics. In a setting where the strong and weak forces play a relevant dynamical role, the framework of classical mechanics itself is typically already faltering and a full extension to quantum mechanics is needed. Hence, our classical mechanics world will deal primarily with gravitational and electromagnetic forces. How about the friction and spring forces encountered in the previous examples, the good old normal force, the tension force in a rope, and a myriad of other force laws that make prominent appearances on the left-hand side of Newton's second law? These are all *macroscopic effective forces*, not fundamental ones. Microscopically, they originate entirely from the electromagnetic force law. For example, when two surfaces in contact rub against each other, the atoms at the interface interact microscopically through Coulomb's law of electrostatics. When we add a large number of these tiny forces, we have an effective macroscopic force that we call **friction**. The microscopic details can be tucked into one single parameter, the coefficient of friction. Similarly, the effect of a large number of liquid molecules on a bacterium average out into a simple force law, $F = -b\, v$, where b is the only parameter left over from the detailed microscopic interactions — which are once again electromagnetic in origin. **Contact forces**, as they are called, are hence approximate statements and originate from the electromagnetic force law.

The reader may rightfully be surprised that complicated microscopic dynamics can lead to rather simple effective force laws — often described by a few macroscopic parameters. This is a rather general feature of the natural laws. When microscopic complexity is averaged over a large number of particles and length scales, it is expected that the resulting macroscopic system is described through simpler laws with fewer parameters. This is not supposed to be obvious, although it may feel intuitive. Realization of its significance and implications in physics underly several physics Nobel prizes in the late twentieth century[2].

[2] The Nobel prize for the development of the renormalization group was awarded to Kenneth G. Wilson in 1982. Wilson described most concisely and elegantly the idea that physics at large length scales is sensitive to physics at small length scales only through a finite number of parameters. However, the idea pervades other major benchmarks of theoretical physics, such as the Nobel prizes of 1999 to Gerardus 't Hooft and Martinus J. G. Veltman and of 1965 to Sin-Itiro Tomonaga, Julian S. Schwinger, and Richard P. Feynman.

1.6 Dimensional analysis

Dimensional reasoning is a powerful tool that can help us learn how one quantity depends upon others. The secret is that in classical mechanics, both sides of an equation must have the same dimensions of mass M, length L, and time T. All other quantities can be expressed in terms of these three. For example, the dimensions of momentum (which we will write as $[p]$, with square brackets) are ML/T, and the dimensions of energy are $[E] = ML^2/T^2$.

For example, suppose we hold up a ball, drop it from rest, and then seek to find its momentum when it strikes the ground. The first step is to ask "what would the momentum likely depend upon?" Using physical intuition, it seems reasonable that the momentum might depend upon the ball's mass m, the height h from which it is dropped, and the acceleration of gravity g. We are not sure *how* it depends upon these quantities, however. The next step is to compare dimensions. The dimensions are $[p] = ML/T, [m] = [M], [g] = L/T^2$, and $[h] = L$. The only way to get the "M" in momentum is to suppose that p is directly proportional to m, because neither g nor h contains a dimension of mass. Then the only way to get the $1/T$ in momentum is to suppose that p is proportional to $\sqrt{g}$. The product $m\sqrt{g}$ has the dimensions $(M/T)\sqrt{L}$, which only needs to be multiplied by $\sqrt{h}$ to achieve the correct dimensions for momentum. That is, the momentum when the ball strikes the ground must have the dependence

$$p = k\, m\sqrt{gh}, \tag{1.79}$$

where k is some *dimensionless* constant. Dimensional reasoning alone cannot give us this constant, so in fact we still do not know what the momentum of the ball is when it reaches the ground. What we do know, however, is that if the momentum at the ground of a particular dropped ball is p_0, the momentum at the ground of a similar ball dropped from twice the height will be $\sqrt{2}\, p_0$, or the momentum of a ball dropped on the Moon from the original height will be $p_0/\sqrt{6}$, since gravity on the Moon is only $1/6^{th}$ that on Earth.

Note that this particular problem is easily solved exactly using $\mathbf{F} = m\,\mathbf{a}$, giving the same equation while providing the value $k = \sqrt{2}$. Dimensional analysis, however, works also in much more complicated problems where the proportionality constant may be more difficult to find.

In general, we can solve such problems by writing a general relation such as

$$p = k\, m^\alpha g^\beta h^\gamma \tag{1.80}$$

where α, β, and γ are constants to be determined. We then expand everything in terms of mass, length, and time

$$\frac{ML}{T} \sim M^{\alpha}\frac{L^{\beta}}{T^{2\beta}}L^{\gamma} \tag{1.81}$$

yielding simple equations for α, β, and γ

$$1 = \alpha \qquad 1 = \beta + \gamma \qquad -1 = -2\,\beta\ , \tag{1.82}$$

confirming that $\alpha = 1$, $\beta = 1/2$, and $\gamma = 1/2$.

EXAMPLE 1-9: Find the rate at which molasses flows through a narrow pipe

By *flow rate*, we means the volume/second (with dimensions [flow rate] = L^3/T) that passes though a pipe. We expect that this depends upon the radius of the pipe, with $[r] = L$, since a wider pipe should allow more fluid to flow than a narrower one. It should also depend upon friction within the fluid itself, and between the fluid and sides of the pipe. Friction in a fluid is characterized by its **viscosity** η, with dimensions $[\eta] = M/LT$, and with values that can be found in tables.[3] The greater the viscosity, the greater the friction, and the lower the flow rate should be: molasses or honey (with high viscosity) should flow more slowly than a light oil (with low viscosity). Finally, the flow rate should also depend upon how hard one pushes on the fluid; *i.e.*, the pressure difference ΔP between one end of the pipe and the other. More precisely, it should depend upon the pressure difference/unit length of pipe, since it makes sense that the viscous friction must be overcome by the pressure gradient within the pipe. The dimensions of pressure are [force/area] = $(ML/T^2)/L^2 = M/(LT^2)$, so the dimensions of pressure per unit length are $[\Delta P/\ell] = M/(L^2T^2)$[4].

[3]The viscosity η of a fluid can be measured in principle by placing the fluid between two parallel metal plates of area A that are separated by a distance d. When one plate is kept fixed while the other is moved parallel to the fixed plate with constant velocity v, the drag force on the moving plate is observed to have the magnitude $F = \eta Av/d$. From this formula one can see that the dimensions of η are M/LT.

[4]In this problem we are assuming smooth, so-called **laminar flow**, which is nonturbulent. High-viscosity fluids (like molasses) that move slowly in narrow pipes are less likely to become turbulent. Turbulent flow is more complicated and depends on additional parameters.

Now we can formally calculate, using dimensional analysis, how the volume per second of the flow depends upon r, η, and $\Delta P/\ell$, by taking arbitrary powers of each and finding the powers by matching dimensions on both sides. That is,

$$\text{flow volume/sec} = k\ r^{\alpha}\eta^{\beta}(\Delta P/\ell)^{\gamma} \tag{1.83}$$

where k is a dimensionless constant. Therefore dimensionally,

$$\frac{L^3}{T} = L^{\alpha}\left(\frac{M}{LT}\right)^{\beta}\left(\frac{M}{L^2T^2}\right)^{\gamma}. \tag{1.84}$$

We match exponents in turn for M, L, and T: That is,

$$\text{mass: } 0 = \beta + \gamma \quad \text{length: } 3 = \alpha - \beta - 2\gamma \quad \text{time:} -1 = -\beta - 2\gamma \tag{1.85}$$

From the first of these we learn that $\gamma = -\beta$, so then from the third equation we find that $\gamma = -\beta = 1$. Finally, the second equation tells us that $\alpha = 3 + \beta + 2\gamma = 4$. Thus the equation for the flow rate through a pipe is

$$\text{flow volume/sec} = k\ \left(\frac{\Delta P/\ell}{\eta}\right) r^4 \tag{1.86}$$

Again, dimensional analysis alone cannot tell us the numerical value of the dimensionless number k. However, we have learned a lot. Most spectacularly, we have learned that the flow rate of a highly viscous fluid is not proportional to the cross-sectional area of the pipe, but to the *fourth power* of the radius: A pipe of twice the radius will transport sixteen times the volume of fluid. This formula corresponds to what is called **Poiseuille flow**, and an exact analytic calculation shows that the constant $k = 6\pi$.

We have carried out the dimensional analysis here in a rather formal way; one can often speed up the process without using arbitrary powers like α, β, and γ. Note from equation (1.86) that the flow rate must depend upon the ratio $(\Delta P/\ell)/\eta$ to cancel out the dimension of mass, so we can rewrite equation (1.86) as

$$\frac{L^3}{T} = L^{\alpha}\left(\frac{M}{L^2T^2} \times \frac{LT}{M}\right)^{\gamma} = L^{\alpha}\left(\frac{1}{LT}\right)^{\gamma}, \tag{1.87}$$

from which it is clear that $\gamma = 1$ to obtain the needed $1/T$ dimension, and so then $\alpha = 4$ to obtain the L^3.

1.7 Synopsis

So much for our very brief summary of Newtonian mechanics. Particles obey Newton's laws of motion, and depending upon the nature of the forces on a particle, one or another of momentum, angular momentum, and energy may be conserved. The momentum of a particle is conserved if there is no net force on it, while the angular momentum of the particle is conserved if there is no net torque on it. Energy is conserved if all the forces acting are conservative and time independent; *i.e.*, if the work done by each force is independent of the path of the particle. Similar laws apply to systems of particles.

Given the forces on a particle together with its initial position and velocity, a classical particle moves along a single, precise path. That is the vision of Isaac Newton: particles follow deterministic trajectories. When viewed from an inertial frame, a particle moves in a straight line at constant speed unless a net force is exerted on it, in which case it accelerates according to $\mathbf{a} = \mathbf{F}/m$.

We have required that the fundamental laws of mechanics obey what is called the principle of relativity, which means that if a law is valid in one inertial frame it is valid in all inertial frames. According to the principle, there is no preferred inertial frame: the fundamental laws can be used by observers at rest in any one of them. This physical statement can be translated into a mathematical statement that given a mathematical transformation of coordinates and other quantities from one frame to another, the fundamental equations should look the same in all inertial frames. We have assumed that the **Galilean transformation** is the correct transformation of coordinates, and have shown that Newton's laws are invariant under that transformation, if any particular force applied is the same in all inertial frames. It is therefore consistent to take Newton's laws as fundamental laws of mechanics.

There is a problem, however. The fundamental laws of **electromagnetism** are *not* Galilean invariant. Therefore something has to give, either the universality of Maxwell's equations of electromagnetism, or the Galilean transformation. In 1905 Albert Einstein decided that it is the Galilean transformation that has to go, which then necessarily compromises our entire understanding of the fundamental laws of mechanics. We will begin to explore the effects of Einstein's ideas in Chapter Two.

1.8 Exercises and Problems

PROBLEM 1-1 : A stream flows at speed $v_W = 0.50$ m/s between parallel shores a distance $D = 35$ m apart. A swimmer swims at speed v_s= 1.00 m/s relative to the water. Use the Galilean velocity transformation to answer the following questions. (a) If the swimmer swims straight toward the opposite shore, i.e., in a direction perpendicular to the shoreline as seen by the swimmer, how long does it take her to reach the opposite shore, and how far downstream is she swept? (b) If instead the swimmer wishes to reach the opposite shore at a spot straight across the stream, at what angle should she swim relative to the stream flow direction, so as to arrive in the shortest time? What is this shortest time?

PROBLEM 1-2 : A river of width D flows at uniform speed V_0. Swimmers A and B, each of whom can swim at speed V_s relative to the water, decide to race one another beginning at the same spot on the shore. Swimmer A swims downstream a distance D relative to the shore, and immediately swims back upstream to the starting point. Swimmer B swims to a point diametrically opposite the starting point on the opposite shore, and then swims back. Assume $v_s > V_0$. Find the total time for each swimmer. Who wins the race?

PROBLEM 1-3 : An ultralight aircraft is 5.0 km due west of the landing field. It can fly 25 km/hr in stationary air. However, the wind is blowing at 25 km/hr from the southwest at a 60^o angle to the direction of the landing field. (a) At what angle to the east must the pilot aim her craft to reach the landing field? (b) How long will it take her to reach the landing field if she flies as described?

PROBLEM 1-4 : A hailstone of mass m is subject to a downward gravitational force mg and an upward force due to air resistance, which we will model here as $F_{\text{drag}} = -kv^2$, where k is a constant and v is the speed of the hailstone relative to the air: the minus sign indicates that the drag force is opposite to the direction of motion. If the model hailstone starts at rest at height h, (a) how long does it take to reach the ground, and (b) what is its speed just before it strikes the ground?

PROBLEM 1-5 : Write out the most general solutions of the (a) overdamped (b) underdamped (c) critically damped harmonic oscillator, expressing in each case the arbitrary constants in terms of the oscillator's initial position x_0 and velocity v_0.

PROBLEM 1-6 : Planets have roughly circular orbits around the Sun. Using the table below of the orbital radii and periods of the inner planets, how does the centripetal acceleration of the planets depend upon their orbital radii? That is, find the exponent n in $a = \text{constant} \times r^n$. (Note that 1 A. U. = 1 astronomical unit, the mean Sun-Earth distance.)

planet	mean orbital radius (A.U.)	period (years)
Mercury	0.387	0.241
Venus	0.723	0.615
Earth	1.000	1.000
Mars	1.523	1.881

PROBLEM 1-7 : A chain is tied tightly between two trees and a force F_0 is applied at right angles to the chain at its midpoint. With the chain in equilibrium, with the ends of the chain at angle θ from the straight line between the trees, what is the tension in the chain?

PROBLEM 1-8 : A rope of mass/length λ is in the shape of a circular loop of radius R. If it is made to rotate about its center with angular velocity ω, find the tension in the rope. *Hint*: consider a small piece of the rope to be a "particle."

PROBLEM 1-9 : One end of a string of length ℓ is attached to a small ball, and the other end is tied to a hook in the ceiling. A nail juts out from the wall, a distance d below the hook. With the string straight and horizontal, the ball is released. When the string becomes vertical it meets the nail, and then the ball swings upward until it is directly above the nail. (a) What speed does the ball have when it reaches this highest point? (b) Find the minimum value of d, as a fraction of ℓ, such that the ball can reach this point at all.

PROBLEM 1-10 : A damped oscillator consists of a mass m attached to a spring k, with frictional damping forces. If the mass is released from rest with amplitude A, and after 100 oscillations the amplitude is $A/2$, what is the total work done by friction during the 100 oscillations?

PROBLEM 1-11 : Half of a chain of total mass M and length L is placed on a frictionless table top, while the other half hangs over the edge. If the chain is released from rest, what is the speed of the last link just as it leaves the table top?

PROBLEM 1-12 : (a) A neutron in a nuclear reactor has a head-on collision with a carbon nucleus, part of the graphite "moderator." The carbon nucleus is initially at rest, and has 12 times the mass of a neutron. What fraction of the neutron's initial speed is lost in the collision? (b) If a neutron collides head-on with a deuteron ($m_d = 2m_n$) used as a moderator in a different reactor, what fraction of the neutron's initial speed is lost? (Slower neutrons are more apt to cause nuclear fission in the fissionable uranium nucleus ^{235}U, and less likely to be lost by absorption in ^{238}U; hence the need for moderators.) Assume elastic (i.e., kinetic-energy conserving) collisions.

PROBLEM 1-13 : Consider an arbitrary power-law central force $\mathbf{F}(\mathbf{r}) = kr^n\hat{\mathbf{r}}$, where k and n are constants and r is the radius in spherical coordinates. Prove that such a force is conservative, and find the associated potential energy of a particle subject to this force.

PROBLEM 1-14 : Estimate the radius (in meters) of the largest spherical asteroid that an astronaut could escape from by jumping.

PROBLEM 1-15 : An overdamped oscillator is released at $x = x_0$ with initial velocity v_0. What is the maximum number of times the oscillator can subsequently pass through $x = 0$?

PROBLEM 1-16 : The potential energy of a mass m on the end of a Hooke's-law spring of force constant k is $(1/2)kx^2$. If the maximum speed of the mass subject to this potential energy is v_0, what are the turning points of the motion?

PROBLEM 1-17 : A simple pendulum is constructed by hanging a bob of mass m on the end of a light cord of length ℓ, pulling it to one side by angle θ_0 from the vertical, and then letting it swing back and forth. We expect that the period P of the pendulum might depend on any or all of m, ℓ, and θ_0, and also the local gravitational field strength g. Using dimensional analysis, find how P depends upon m, ℓ, and g. (The angular amplitude θ_0 is dimensionless, so we cannot learn how P depends on θ_0 using dimensional analysis alone.)

PROBLEM 1-18 : The velocity of waves on the surface of a lake depends upon gravity g and the depth h of the lake, as long as the wavelength of the waves satisifies $\lambda \gg h$, corresponding to what is called "shallow water waves". If a traveling wave has velocity v_0 and subsequently encounters a part of the lake that is twice as deep, by what factor will the wave velocity be changed?

PROBLEM 1-19 : The velocity of waves on the surface of a lake depends upon gravity g and the wavelength λ as long as the depth of the lake h satisfies $h >> \lambda$, corresponding to what is called "deep water waves". If we were to increase the wavelength by a factor of two, by what factor would the wave velocity be changed?

PROBLEM 1-20 : Capillary waves on the surface of a liquid come about because of the liquid's surface tension σ, which has dimensions M/T^2. The velocity of capillary waves depends upon σ and also upon the wavelength λ and the density ρ of the liquid. Two capillary waves on the same liquid have wavelengths λ_1 and $\lambda_2 = 2\lambda_1$. What is the ratio of their velocities?

PROBLEM 1-21 : The **Planck length** ℓ_p depends upon Planck's constant $\hbar$, Newton's constant of gravity G, and the speed of light c. If Planck's constant were twice as large as it actually is, how would that affect ℓ_p? How would it affect the **Planck time** t_p and **Planck mass** m_p, also both defined in terms of the same three fundamental constants? Taking the proportionality constant to be unity in each case (which is how they are actually defined), how large are the Planck length, mass, and time numerically in SI units (kilograms, meters, seconds)?

PROBLEM 1-22 : Two very flat parallel metal plates, with a vacuum between them and surrounding them, are attracted to one another by what is called the Casimir force, as predicted by quantum field theory. This force is proportional to the area A of each plate, and also depends upon the distance d between the plates, the speed of light c, and Planck's constant (divided by 2π) $\hbar$. If the distance d is reduced by half, does the Casimir force increase or decrease? By what factor?

PROBLEM 1-23 : A ball of mass m is tossed straight upward from the ground with velocity v_0. The time it takes for it to rise and fall back to the ground might depend upon its mass m, v_0, and the gravitational field g. If v_0 were doubled, by what factor would the time above ground increase? What would happen in m were doubled, keeping v_0 and g the same?

PROBLEM 1-24 : Two astronauts are instantaneously at rest above a spherical asteroid of mass m and radius R. One astronaut is at distance r and the other at distance $2r$ from the asteroid's center, where $r \ll R$. If it takes time T_0 for the first astronaut to fall to the asteroid, about how long does it take the second to fall?

PROBLEM 1-25 : An exploding nuclear bomb creates a rapidly-expanding shock wave in the air surrounding the blast. Within the shock wave the air glows brightly, because it has been strongly heated, giving the appearance of a fireball. The radius R of the expanding ball of hot air depends upon time t, the energy E of the blast, and the density ρ of the air. (It might depend also upon the ambient air pressure, but to a good approximation it does not, because the ambient pressure is so much smaller than the pressures created by the blast.) (a) Using dimensional reasoning, find out how R depends upon t, E, and ρ. (b) If the shock wave has radius R_0 at time 0.01 s, what is its radius at 0.1 s? (c) The picture shows the fireball of the Trinity test, the first nuclear explosion, at 05.30 hours, 16 July 1945, Alamogordo, N. M., at time 0.025 s after detonation. The diameter is about 250 m, as shown. Estimate the energy of the blast in joules and in equivalent tonnes of TNT, assuming the dimensionless coefficient in the expression for $R(t, E, \rho)$ is unity (the coefficient has been calculated to be 1.003). The explosive energy of one ton of TNT is 4.2×10^9 J. [Pictures of the first explosion were published in *Life Magazine*. Using dimensional reasoning, several physicists around the world deduced the yield of the explosion.] (INCLUDE PHOTO)

PROBLEM 1-26 : Steady rain falls at constant speed v^r straight down as observed by a pedestrian standing on a sidewalk. A bus travels along the horizontal street at speed v_b. (a) At what angle θ to the vertical do the raindrops fall, as seen by the bus driver? (b) Suppose the bus has no windshield, leaving a hole in the flat, vertical front of the bus. The driver wants to travel forward at constant speed in a straight line from point A to point B. To minimize the total amount of water entering through the hole, should the driver drive the bus very slowly, as quickly as possible, or how?

PROBLEM 1-27 : An object of mass m is subject to a drag force $F = -kv^n$, where v is its velocity in a medium, and k and n are constants. If the object starts with velocity v_0 at time $t = 0$, find its subsequent velocity as a function of time.

PROBLEM 1-28 : A space traveler pushes off from his coasting spaceship with relative speed v_0; his mass and spacesuit have mass M, and he is carrying a wrench of mass m. Twenty minutes later he decides to return, but his thruster doesn't work. In another forty minutes his oxygen supply will be exhausted, so he immediately throws the wrench away from the ship at speed v_w relative to himself prior to the throw. (a) What then is his speed relative to the ship? (b) In terms of given parameters, what is the minimum value of v_w needed so he will return in time?

PROBLEM 1-29 : A neutron of mass m and velocity v_0 collides head-on with a ${}^{235}_{92}U$ isotope of mass M at rest in a nuclear reactor, and the neutron is absorbed to form ${}^{236}_{92}U$. (a) Find the velocity v_A of the ${}^{236}_{92}U$ isotope in terms of m, M, and v_0. (b) The ${}^{236}_{92}U$ isotope subsequently fissions into two isotopes of equal mass, each emerging at angle θ to the forward direction. Find the speed v_B of each final isotope in terms of given parameters.

PROBLEM 1-30 : It typically takes about 10 months for a spacecraft journey from Earth to Mars, and because of the loss of bone mass and other physiological problems it may be worthwhile providing an artificial gravity for humans to make the trip. One proposal is to attach the spacecraft (of mass M) to one end of a straight cable of length ℓ , attach an equal-mass counterweight to the other end, and then make the entire assembly rotate about the center of the cable with angular velocity ω. (a) Find the effective gravity within the spacecraft in terms of given parameters. (b) If the cable has negligible mass, find the tension within it as a function of the distance r from its center. (c) If instead the cable has constant mass per unit length λ, find the tension within this cable as a function of r.

PROBLEM 1-31 : A circular hoop of wire of radius R is oriented vertically, and is then forced to rotate with angular velocity ω about a vertical axis through its center. A small bead of mass m slides frictionlessly on the hoop. There is a downward gravitational field g. (a) Show that if $\omega > \sqrt{g/R}$, there are four different locations on the hoop for which the bead can be in equilibrium. Find the angles θ for these four locations, where θ is the angle of the point up from the bottom of the hoop, measured between the vertical and a radial line from the center of the hoop. (b) Show that if $\omega < \sqrt{g/R}$, there are only two equilibrium positions. Where are they?

PROBLEM 1-32 : A particle of mass m is subject to the force $F = \alpha \sin(kx)$. (a) If the maximum value of the corresponding potential energy is α/k, what are the turning points for a particle of energy $E = \alpha/2k$? (b) Find the speed of the particle as a function of position, if the particle starts at rest at one of the turning points. (c) Find an expression for the position of the particle as a function of time.

PROBLEM 1-33 : Show that if a mass distribution is spherically symmetric, the gravitational field inside it is directed radially inward, and its magnitude at a radius r from the center is simply $GM(r)/r^2$, where $M(r)$ is the mass within the sphere whose radius is r.

PROBLEM 1-34 : A non-rotating uniform-density spherical asteroid has mass M and radius R. (a) If a straight tunnel is drilled through the asteroid from one side to the other, which passes through the asteroid's center, how long would it take an astronaut to fall from one end of the tunnel to the other and back to the starting point again, by simply stepping into the tunnel at one end? (b) If a different straight tunnel is drilled through the same asteroid, where this time the tunnel misses the asteroid's center by a distance $R/2$, how long would it take the astronaut to fall from one end to the other and back, assuming there is no friction between the sides of the tunnel and the astronaut? (c) Now suppose that instead of falling through the tunnel, the astronaut is given an initial tangential velocity of just the right magnitude so the astronaut is inserted into circular orbit just above the surface. How long will it take the astronaut to return to the starting point in this case?

PROBLEM 1-35 : Four mathematically equivalent conditions for a force to be conservative are given in the chapter. Select one of the conditions and suppose that it is valid for some force $\mathbf{F}$. Show that each of the other three conditions is a necessary consequence.

PROBLEM 1-36 : A particle is attached to one end of an unstretched Hooke's-law spring with force constant k. The other end of the spring is fixed in place. If now the particle is pulled so the spring is stretched by a distance x, the potential energy of the particle is $U = (1/2)kx^2$. (a) Now suppose there are *two* unstretched springs with the same force constant k that are laid end-to-end in the y direction, with a particle attached between them. The other ends of the springs are fixed in place. Now the particle is pulled in the transverse direction a distance x. Find its potential energy $U(x)$. (b) $U(x)$ is proportional to what power of x for small x, and to what power of x for large x?

PROBLEM 1-37 : A particle of mass m is subject to the central attractive force $\mathbf{F} = -k\mathbf{r}$, a force that in effect is that of a Hooke's-law spring of zero unstretched length, whose other end is fixed to the origin. The particle is placed at an initial position $\mathbf{r}_0$ and then given an initial velocity $\mathbf{v}_0$ that is not colinear with $\mathbf{r}_0$. (a) Explain why the subsequent motion of the particle is confined to a plane that contains the two vectors $\mathbf{r}_0$ and $\mathbf{v}_0$. (b) Find the potential energy of the particle. (c) Explain why the particle's angular momentum is conserved about the origin, and use this fact to find a first-order differential equation of motion involving r and dr/dt. (d) Solve the equation for $t(r)$, and show that the particle has both an inner and an outer turning point.

PROBLEM 1-38 : A rock of mass m is thrown radially outward from the surface of a spherical, airless moon. From Newton's second law its acceleration is $\ddot{r} = -GM/r^2$, where M is the moon's mass and r is the distance from the moon's center to the rock (the minus sign indicates that the acceleration is inward, toward the moon's surface). The energy of the rock is conserved, so $(1/2)m\dot{r}^2 -$

$GMm/r = E =$ constant. (a) Show by differentiating this latter equation that energy conservation is a first integral of $F = m\ddot{r}$ in this case. (b) What is the minimum value of E (in terms of given parameters), for which the rock will escape from the moon? (c) For this case of the escape energy E_{esc}, what is $\dot{r}(t)$, the velocity of the rock as a function of the time since it was thrown? Also find $\dot{r}(t)$ if (d) $E > E_{esc}$ (e) $E < E_{esc}$.

PROBLEM 1-39 : The Friedman equations have played an important role in big-bang cosmology. They feature an "expansion factor" $a(t)$, proportional to the distance between any two points (such as the positions of two galaxies) that are sufficiently remote from one another that local random motions can be ignored. If a increases with time, the distance between galaxies increases proportionally, corresponding to an expanding universe. If we model for simplicity the universe as filled with pressure-free dust of uniform density ρ, the Friedman equations for $a(t)$ are

$$\ddot{a} = -\frac{4\pi G\rho}{3}a \quad \text{and} \quad \dot{a}^2 = \frac{8\pi G\rho}{3}a^2 - \frac{kc^2}{R_0^2} \tag{1.88}$$

where G is Newton's gravitational constant, c is the speed of light, R_0 is the distance between two dust particles at some particular time t_0, and $k = +1$, -1, or 0. The density of the dust is inversely proportional to the cube of the scale factor $a(t)$, i.e., $\rho = \rho_0(a_0/a)^3$, where ρ_0 is the density when $a = a_0$. Therefore

$$\ddot{a} = -\frac{4\pi G\rho_0 a_0^3}{3a^2} \quad \text{and} \quad \dot{a}^2 = \frac{8\pi G\rho_0 a_0^3}{3a} - \frac{kc^2}{R_0^2}. \tag{1.89}$$

(a) Show that if we set the origin to be at one of the two chosen dust particles, then if M is the total mass of dust within a sphere surrounding this origin out to the radius of the other chosen particle, then the equations can be written

$$\ddot{a} = -\frac{(GM/R_0^3)}{a^2} \quad \text{and} \quad \frac{1}{2}\dot{a}^2 - \frac{(GM/R_0^3)}{a} = -\frac{kc^2}{2R_0^2} \equiv \epsilon \tag{1.90}$$

where ϵ and M are constants. (b) Show that the second equation is a first integral of the first equation. (c) Compare these equations to the $F = ma$ and energy conservation equations of a particle moving radially under the influence of the gravity of a spherical moon of mass M. (d) Einstein hoped that his general-relativistic equations would lead to a static solution for the universe, since he (like just about everyone before him) believed that the universe was basically at rest. The Friedman equations resulting from his theory show that the universe is generally expanding or contracting, however, just as a rock far from the Earth is not going to stay there, but will generally be either falling inward or on its way out. So Einstein modified his theory with the addition of a "cosmological constant" Λ, which changed the Friedman equations for pressure-free dust to

$$\ddot{a} = -\frac{(GM/R_0^3)}{a^2} + \frac{\Lambda}{3}a \quad \text{and} \quad \frac{1}{2}\dot{a}^2 - \frac{(GM/R_0^3)}{a} - \frac{\Lambda}{6}a^2 = \epsilon. \tag{1.91}$$

Show that these equations *do* have a static solution, and find the value of Λ for which the solution is static. (e) Show however (by sketching the effective potential energy function in the second equation) that the static solution is *unstable*, so that if the universe is kicked even slightly outward it will accelerate outward, or if it is kicked even slightly inward it will collapse. A static solution is therefore physically unrealistic. (Einstein failed to realize that his static solution was unstable, and later, when Edwin Hubble showed from his observations at the Mount Wilson Observatory that the universe is in fact expanding, Einstein declared that introducing the cosmological constant was "my biggest blunder".) (f) Suppose the cosmological constant is retained in the equations, but that the dust is removed so that $M = 0$. Solve the equations for $a(t)$ in this case. The solution is the **de Sitter model**, an "inflationary" model of the expanding universe. What is the constant ϵ for the de Sitter model? (g) Make a qualitative sketch of $a(t)$ if both M and Λ are nonzero. Of the terms containing M and Λ, which dominates for small times? For large times?