Writing DEs with Dimensionless Variables

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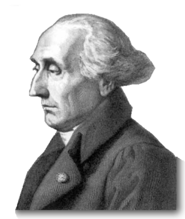
Physics 111



When studying the behavior of a mechanical system, it is often useful to rewrite the equations in dimensionless form, parameterized by a number of dimensionless quantities.

The Lagrange equation(s) for your project will involve variables (e.g., x, t, etc.) with dimensions and also parameters (e.g., g, L, R, Ω, etc.) with dimensions. Before solving for the motion using numerical methods, it is helpful to rewrite the equations in dimensionless form. Here's how.

Consider the pendulum equation,

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

where L is the length of the massless rod. The variable θ is already dimensionless, but t has dimensions of time, and the parameters g and L both have dimensions as well. Let $t = CT$, where C is a constant (with dimensions of time) and T is a dimensionless time variable. Substituting into the differential equation, we find

$$C^{-2}\frac{d^2\theta}{dT^2} + \frac{g}{L}\sin\theta = 0$$

Now choose $C^{-2} = g/L$ so the equation becomes

$$\boxed{\frac{d^2\theta}{dT^2} + \sin\theta = 0}$$

What's the advantage? You can now solve this equation (numerically, if necessary) once and for all with given initial conditions $\theta(0)$ and $\frac{d\theta}{dT}(0)$. It is not necessary to choose g and L and then repeat the calculations for different values of g and L. Furthermore, the dimensionless variable T is more likely to involve moderate-size numbers than the dimensional variable t, which helps avoid numerical errors and makes intermediate values easier to understand when debugging.

Now consider a slightly more complicated differential equation:

$$m\ddot{x} + b\dot{x} + kx = F\cos\omega t \implies \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F}{m}\cos\omega t$$

where I have defined $2\beta = b/m$ and $\omega_0^2 = k/m$. There are now two obvious frequencies in the problem: the natural frequency ω_0 and the drive frequency ω. Clearly, the product of one of these with the time variable will give a dimensionless time variable we can use. In this case, I'd propose defining $T \equiv \omega_0 t$ and $\Omega \equiv \omega/\omega_0$. However, our work is not done yet. On substituting, we get

$$\omega_0^2 x'' + 2\beta\omega_0 x' + \omega_0^2 x = \frac{F}{m}\cos\Omega T$$

where primes indicate differentiation with respect to the dimensionless time variable T. Clearly, we can divide out ω_0^2 on the left-hand side, giving an expression that has dimensions of length. To get rid of the length, let us adopt $F/m\omega_0^2$ as the scale for x by setting $x = XF/m\omega_0^2$. Furthermore, define the dimensionless frequency ratio $\beta/\omega_0 \equiv B$. Putting this all together gives

$$X'' + 2BX' + X = \cos\Omega T$$

which clearly shows that there are two dimensionless parameters, B and Ω, that define the class of solutions.