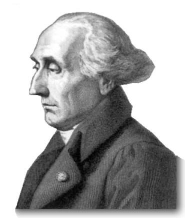
Lightning Overview of Contour Integration

Wednesday, 19 October 2011

Physics 111



The prettiest math I know, and extremely useful.

1. Overview

This handout summarizes key results in the theory of functions of a complex variable, leading to the inverse Laplace transform. I make no pretense at rigor, but I'll do my best to ensure that I don't say anything false. The main points are:

1. The Cauchy-Riemann conditions, which must be satisfied by a differentiable function $f(z)$ of a complex variable z.
2. $\oint f(z)\,dz = 0$ if $f(z)$ is continuous and differentiable everywhere inside the closed path.
3. If $f(z)$ blows up inside a closed contour in the complex plane, then the value of $\oint f(z)\,dz$ is $2\pi i$ times the sum of the residues of all the poles inside the contour.
4. The Fourier representation of the delta function is $\int_{-\infty}^{\infty} e^{ikx}\,dk = 2\pi\delta(x)$.
5. The inverse Laplace transform equation.

2. Cauchy-Riemann

For the derivative of a function of a complex variable to exist, the value of the derivative must be independent of the manner in which it is taken. If $z = x + iy$ is a complex variable, which we may wish to represent on the xy plane, and $f(z) = u(x,y) + iv(x,y)$, then the value of $\frac{df}{dz}$ should be the same at a point z whether we approach the point along a line parallel to the x axis or a line parallel to the y axis. Therefore,

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial y} \qquad \text{(along } x\text{)}$$

$$\frac{df}{dz} = \frac{\partial u}{\partial(iy)} + i\frac{\partial v}{\partial(iy)} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \text{(along } y\text{)}$$

Comparing real and imaginary parts, we obtain the Cauchy-Riemann equations:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \tag{1}$$

3. Integrals around closed paths

Stokes's theorem holds that $\oint \mathbf{F}\cdot d\mathbf{l} = \oiint (\nabla\times\mathbf{F})\cdot d\mathbf{A}$, indicating that the integral of the curl of a vector function over an enclosed area is equal to the line integral of the function over the boundary to the area. The right-hand rule is used to orient the contour with respect to the outward normal to the area.

The z component of the curl is $(\nabla \times \mathbf{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$. By Stokes's theorem, the integral of this quantity over a closed area in the xy plane must be equal to the line integral around the area,

$$\iint \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx\,dy = \oint (F_x\,dx + F_y\,dy)$$

Now consider the integral of a function of a complex variable around a closed contour on the complex plane,

$$\oint f(z)\,dz = \oint (u + iv)(dx + i\,dy) = \oint (u\,dx - v\,dy) + i \oint (u\,dy + v\,dx)$$

We can now use Stokes's theorem to show that each of the integrals on the right vanishes. For the first integral, let $F_x = u$ and $F_y = -v$. Then

$$\oint (u\,dx - v\,dy) = \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx\,dy$$

but the integrand vanishes by the Cauchy-Riemann equation. Similarly for the second integral, let $F_x = v$ and $F_y = u$. Then

$$\oint (v\,dx + u\,dy) = \iint \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx\,dy = 0$$

Therefore, **the integral of $f(z)$ around a closed contour vanishes, provided that $f(z)$ is everywhere differentiable inside the contour.**

4. Residues

Suppose that $f(z)$ blows up at a point z_0 inside a closed contour. Although I will not justify this, suppose further that in the neighborhood of z_0 the function may be expanded in a **Laurent series**, which is a generalization of Taylor's series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

Notice that unlike Taylor's series, the Laurent series includes both negative and positive powers. Since the integral around any closed contour that doesn't contain a singularity vanishes, we are free to deform the contour to include a detour that comes within ϵ of z_0, then goes around a circle of radius ϵ centered on z_0 and then heads right back out from where it came, as illustrated in Fig. 1. The contributions along the path in and then out precisely cancel, and we are left with the contribution around the small circle.

Let $\zeta = z - z_0 = \epsilon e^{i\theta}$, where ϵ is a constant. Then

$$\oint f(z)\,dz = \oint \left(\sum_{n=-\infty}^{\infty} a_n \zeta^n\right) d\zeta = \sum_{n=-\infty}^{\infty} \oint a_n \zeta^n\,d\zeta$$

where I have blithely interchanged the order of integration and summation. See the math department for the details. We now must evaluate $\oint \zeta^n\,d\zeta$, but unless $n = -1$, $\int \zeta^n\,d\zeta = \frac{\zeta^{n+1}}{n+1}$. This is a single-valued function of ζ, which means that when we evaluate at $\zeta = \epsilon e^{i\theta}$ for $\theta = 2\pi$ and $\theta = 0$, we get the same value. Hence, the integral vanishes.

When $n = -1$ we have to proceed more carefully:

$$\oint \frac{d\zeta}{\zeta} = \ln(\epsilon e^{i\theta})\Big|_0^{2\pi} = (\ln\epsilon + i\theta)\Big|_0^{2\pi} = 2\pi i \qquad (2)$$

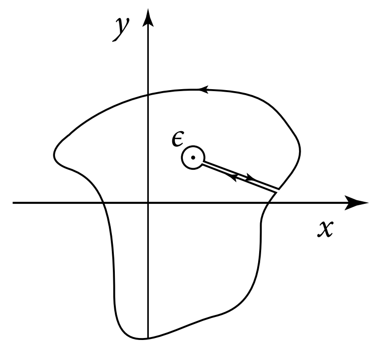


Figure 1: *Deforming a contour to remove a pole from the interior of a region. Only the $a_{-1}\zeta^{-1}$ term in the Laurent expansion of $f(z)$ contributes to the integral around the circle of infinitesimal radius ϵ; the coefficient a_{-1} is called the* residue *of the pole.*

So, the only term in the infinite series that contributes is the one with $n = -1$, and it contributes $2\pi i$ times the coefficient a_{-1}. This coefficient is called the **residue**. Thus, we have the **residue theorem**:

$$\oint f(z)\,dz = 2\pi i \sum \text{residues} \qquad (3)$$

where the sum of residues means the sum of the residues of all poles lying within the contour. If the contour runs through a pole, then we get half the residue, as you can readily verify by returning to Eq. (2) and noting that we now integrate only through an angular range of π.

5. Fourier representation of $\delta(x)$

We now seek to establish that

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk \qquad (4)$$

where $\delta(x)$ is the Dirac delta "function." First a plausibility argument. If we use Euler's theorem to express the complex exponential as $e^{ikx} = \cos kx + i \sin kx$, then we see immediately that the imaginary part must vanish because $\sin kx$ is an odd function of k and we are integrating over the infinite interval. All the cosines peak at the origin and then proceed to oscillate at different spatial frequencies as we move away. These oscillations tend to wipe one another out at any nonzero value of x, but they all add up at $x = 0$ to produce an enormous (infinite) spike. With only a tiny bit of pixie dust to supply the factor of $1/2\pi$, we can see how the integral might just produce $\delta(x)$.

Of course, whenever you see a delta function you feel an Pavlovian urge to integrate. I know I do. So consider

$$I = \int_{x_1}^{x_2}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk\right) dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{x_1}^{x_2} e^{ikx}\,dx$$

where I have once again blithely interchanged the order of integration. If Eq. (4) holds, then $I = 0$ if x_1 and x_2 don't bracket 0, and $I = 1$ if $x_2 > 0$ and $x_1 < 0$. We evaluate I in two steps. The first integral is child's play:

$$\int_{x_1}^{x_2} e^{ikx}\,dx = \left.\frac{e^{ikx}}{ik}\right|_{x_1}^{x_2} = \frac{e^{ikx_2}}{ik} = \frac{e^{ikx_2} - e^{ikx_1}}{ik}$$

Now we have to evaluate integrals of the form

$$J(x) = \int_{-\infty}^{\infty} \frac{e^{ikx}}{ik}\,dk \tag{5}$$

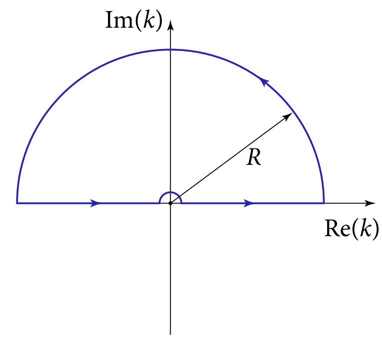


Figure 2: *Contour to evaluate Eq. (5).*

in which the integrand diverges right in the middle of the path of integration. If we expand the exponential in a series $e^{ikx} = 1 + ikx + \frac{(ikx)^2}{2!} + \cdots$, we can readily see that the coefficient of the a_{-1} term is just $1/i$. Have no fear. There are two possibilities to worry about: $x > 0$ and $x < 0$. If $x > 0$, then we can create a closed contour on the complex k plane using a semicircular arc of radius $k = R$ in the upper half plane, as illustrated in Fig. 2, and take the limit as $R \to \infty$. All along this arc, the imaginary part of the complex variable k is positive and enormous, so the semicircular integral contributes nothing in the limit as $R \to \infty$. By the residue theorem, then, $J(k)$ is equal to $2\pi i$ times the residue of the single pole at $k = 0$, except that this pole lies right on the contour of integration. By the argument leading to Eq. (2), we get half the residue. Hence,

$$J(x) = \frac{1}{i}(\pi i) = \pi, \qquad x > 0$$

If $x < 0$, then we must close in the lower half plane and we now traverse the closed contour in the negative (clockwise) direction. Now we get minus half the pole, so

$$J(x) = \frac{1}{i}(-\pi i) = -\pi, \qquad x < 0$$

Putting this all together, we have

$$I = \frac{1}{2\pi}\left[J(x_2) - J(x_1)\right] = \begin{cases} 1 & x_1 < 0 < x_2 \\ -1 & x_2 < 0 < x_1 \\ 0 & \text{otherwise} \end{cases}$$

This confirms Eq. (4).

Exercise 1 One definition of the Fourier transform of a function $f(t)$ (assumed to die off appropriately as $|t| \to \infty$) is

$$\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} f(t) e^{i\omega t}\,dt \tag{6}$$

Given $\tilde{f}(\omega)$, show that the original function $f(t)$ may be recovered by computing the inverse transform, given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t}\, d\omega \tag{7}$$

Caution: take care not to confuse dummy variables of integration with "real" variables.

6. Laplace Transforms

Some dynamics problems may be solved using Laplace transforms to convert differential equations to algebraic ones. The Laplace transform of a function $g(t)$ is defined by

$$\mathscr{G}(s) \equiv \int_0^{\infty} g(t) e^{-st}\, dt \equiv \mathcal{L}\{g(t)\} \tag{8}$$

where we require that $g(t) = 0$ for $t < 0$. (You can always shift the origin of time if your function starts up at some distinct value of t less than 0.) Note that as s gets large, only the portions of $g(t)$ at small t contribute to $\mathscr{G}(s)$. You can readily confirm that the Laplace transform of $g(t) = e^{-\alpha t}$ is $\mathscr{G}(s) = \frac{1}{\alpha+s}$, and that the Laplace transform of $g(t) = \sin \omega t$ is $\mathscr{G}(s) = \frac{\omega}{\omega^2+s^2}$. (You'll need to do two integrations by parts for this one.)

Lots and lots of Laplace transforms are tabulated in books; engineers especially love them. Typically, they solve a problem by finding the Laplace transform of the function they really want and then have to go backwards to find that function. One approach is to look in the book of transforms until you find the one that matches your transform and then copy down the corresponding function that produces this transform. This is sort of like figuring out the integral of a function f by taking the derivative of all sorts of functions until you find a derivative that matches f. Eventually, you'd like to learn how to integrate!

So, our problem is, given a function $\mathscr{G}(s)$ that is defined for $s \geq 0$ and which doesn't blow up as $s \to \infty$, how do we find the function $g(t)$ for which $\mathscr{G}(s)$ is the Laplace transform?

From the previous section, we know that $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dx$. If we could somehow turn the e^{-st} into a delta function with argument $t - t'$, then we could pick off the value of $g(t_0)$. So, let's multiply $\mathscr{G}(s)$ by $e^{st'}$ and integrate:

$$\int \mathscr{G}(s) e^{st'}\, ds = \int \int_0^{\infty} g(t) e^{-st}\, dt\, e^{st'}\, ds = \int \int_0^{\infty} g(t) e^{-s(t-t')}\, dt\, ds \tag{9}$$

I've been a bit coy about the limits of integration for s. If we want to use the Fourier representation of the delta function, then we have to change s into is somehow. Suppose we integrate along the line $s = b + iy$, where b is a real constant that puts the line, shown in Fig. 3, to the right of any poles in $\mathscr{G}(s)$. Then we could close the contour using the arc at infinity shown in the dotted line without changing the value of the integral, since the $e^{st'} e^{bt'}$ term will cause it to vanish for large negative values of s. Then

$$\int_{b-i\infty}^{b+i\infty} \mathscr{G}(s) e^{st'}\, ds = \int_{b-i\infty}^{b+i\infty} \int_0^{\infty} g(t) e^{s(t'-t)}\, dt\, ds \tag{10}$$

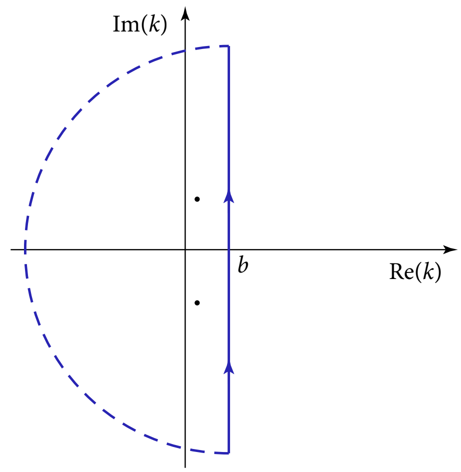


Figure 3: *Contour for evaluating the inverse Laplace transform.*

Now we blithely interchange the order of integration on the right (acceptable, provided that $g(t)$ is well behaved at infinity) to get

$$\begin{aligned}\int_{b-i\infty}^{b+i\infty} \mathscr{G}(s) e^{st'}\,ds &= \int_0^\infty g(t) \int_{b-i\infty}^{b+i\infty} e^{s(t'-t)}\,ds\,dt \\ &= \int_0^\infty g(t) e^{b(t'-t)} \left[\int_{-\infty}^{\infty} e^{iy(t'-t)} i\,dy\right] dt \end{aligned} \tag{11}$$

The term in brackets is just $2\pi i \delta(t'-t)$, which makes the integral over t straightforward. We get

$$\int_{b-i\infty}^{b+i\infty} \mathscr{G}(s) e^{st'}\,ds = 2\pi i g(t') \tag{12}$$

Changing t' to t and tidying up, we have the Laplace transform inversion formula,

$$\boxed{g(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} \mathscr{G}(s) e^{st}\,ds} \tag{13}$$

where the value of b must be chosen to be greater than the real coordinate of any poles in $\mathscr{G}(s)$. Eq. (13) is sometimes called the **Bromwich integral**. By the residue theorem, therefore, the inverse transform is just the sum of the residues of $\mathscr{G}(s)e^{st}$.

Example 1: Bromwich integral

Given the Laplace transform $\mathscr{G}(s) = \dfrac{s+a}{(s+a)^2+k^2}$, find $g(t)$.

The poles of $\frac{s+a}{(s+a)^2+k^2}e^{st}$ occur where $s+a = \pm ik$, so let $s = -a \pm ik + z$ and rewrite the integrand in terms of the new variable z:

$$\mathscr{G}(s)e^{st} = \frac{\pm ik + z}{(\pm ik + z)^2 + k^2}e^{(\pm ik - a + z)t} = \frac{\pm ik + z}{\pm 2ikz + z^2}e^{(\pm ik - a + z)t} \tag{14}$$

We can now take the limit as $z \to 0$ and look for the term proportional to z^{-1}; we get

$$a_{-1}^{\pm} = \frac{1}{2}e^{\pm ikt - at} \tag{15}$$

Summing the residues, we find

$$g(t) = \frac{1}{2}e^{-at}\left(e^{ikt} + e^{-ikt}\right) = e^{-at}\cos kt \tag{16}$$

You can confirm that $\mathcal{L}\left\{e^{-at}\cos kt\right\}$ is indeed $\frac{s+a}{(s+a)^2+k^2}$.

7. Exercises and Problems

Exercise 2 Show that the Laplace transform of the derivative of a function $g(t)$ is

$$\mathcal{L}\left\{\frac{dg}{dt}\right\} = s\mathscr{G}(s) - g(0) \tag{17}$$

where $\mathscr{G}(s) = \mathcal{L}\{g(t)\}$. In other words, the Laplace transform converts derivatives to powers of s.

Exercise 3 The convolution of two functions is defined by

$$f(t) = \int_0^t g_1(t-t')g_2(t')\,dt' \tag{18}$$

Show that $\mathcal{L}\{f(t)\} = G_1(s)G_2(s)$.

Exercise 4 Calculate the Laplace transform of $\sin\omega t$.