

Accelerated Reference Frames (revised)

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Newton's equations apply in inertial frames, but sometimes it is more convenient to work in an accelerated frame. In accelerated frames, pseudo forces corresponding to the acceleration of the frame must be added to provide a consistent description of the dynamics.

Physics 111



Center of Mass

Before turning to accelerated frames, I will reiterate a point made before about the kinetic energy of a system of mass points. Namely, the kinetic energy may be expressed in terms of the motion of the center of mass and the motion with respect to the center of mass. The derivation is important but not difficult, so here goes.

The kinetic energy of the particles is

$$T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}'_{\alpha} \cdot \dot{\mathbf{r}}'_{\alpha} = \frac{1}{2} \sum_{\alpha} m_{\alpha} (\dot{\mathbf{R}} + \dot{\mathbf{r}}_{\alpha}) \cdot (\dot{\mathbf{R}} + \dot{\mathbf{r}}_{\alpha})$$

where $\mathbf{r}'_{\alpha} = \mathbf{R} + \mathbf{r}_{\alpha}$ is the position of the α th particle with respect to an inertial frame,

$$\mathbf{R} \equiv \frac{1}{M} \sum_{\alpha} m_{\alpha} \mathbf{r}'_{\alpha} \quad (1)$$

is the center of mass of the particles, $M = \sum_{\alpha} m_{\alpha}$, and \mathbf{r}_{α} is the position of the α th particle *with respect to the center of mass*. Expanding the dot product gives

$$T = \frac{1}{2} \sum_{\alpha} (m_{\alpha} \dot{\mathbf{R}}^2 + m_{\alpha} \dot{\mathbf{r}}_{\alpha}^2 + 2m_{\alpha} \dot{\mathbf{R}} \cdot \dot{\mathbf{r}}_{\alpha}) = \frac{1}{2} \left(M \dot{\mathbf{R}}^2 + \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha}^2 + \dot{\mathbf{R}} \cdot \frac{d}{dt} \sum_{\alpha} m_{\alpha} \mathbf{r}_{\alpha} \right)$$

but by the definition of the center of mass in Eq. (1), the sum in the final term in parentheses vanishes. Therefore,

$$T = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_{\alpha} m_{\alpha} \dot{\mathbf{r}}_{\alpha}^2 \quad (2)$$

which is to say that the kinetic energy may be expressed as that of a single mass point having the entire mass of the system, located at the center of mass, and moving with the center of mass, plus the kinetic energy of particles moving with respect to the center of mass. We shall see shortly that the summation in Eq. (2) takes a simpler form in the case of a (rotating) rigid body, but the expression given here is perfectly general.

Exercise 1 Show that if the particles interact amongst themselves via forces that obey Newton's third law, in addition to any external forces, the acceleration of the center of mass is equal to the sum of all external forces divided by the total mass of the particles.

1. Accelerated Frames

Let's remind ourselves of Newton's three laws, first in Motte's 1729 translation:

- I. *Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impress'd thereon.*
- II. *The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress'd.*
- III. *To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.*

The first law claims that in the absence of external forces, a body either remains at rest or it moves at constant velocity. In both cases, the claim makes sense only if we are looking at the body in an inertial frame of reference. Put another way, the first law argues that there are inertial frames of reference and motion is particularly dull in such frames for objects free from external forces.

The second law needs a little translational help. If motion simply means velocity, then the (time rate of) change in velocity is proportional to the impressed force; on the other hand, if we understand *motion* to mean momentum, then we have

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \quad (3)$$

In either case, the statement presumes that we are looking at the "body" from an inertial frame of reference.

The third law argues that forces come in pairs: the force on A caused by B is equal and opposite to the force on B caused by A. Their vector sum is always zero.

Newton's laws hold (at least approximately) in inertial reference frames. We have built Lagrangian and Hamiltonian dynamics on the Newtonian bedrock, so

you ought not to be surprised that they presume that coordinates are measured with respect to an inertial frame. Sometimes, however, that's just not particularly convenient. If we happened to be doing physics inside an elevator—and frankly, who doesn't from time to time?—then we might find it much more convenient to use the elevator as the reference frame and patch up Newton's laws by adding in one or more terms to account for the motion of the noninertial (elevator) frame. In general, such a frame may both translate and rotate with respect to an inertial frame. How can we account for this?

Translations are straightforward. Let the coordinates of a point on the elevator be \mathbf{R} and let us measure positions within the elevator with respect to said point. Then

$$\mathbf{r}' = \mathbf{R} + \mathbf{r}$$

where \mathbf{r}' is the position of a mass point with respect to an inertial frame, \mathbf{R} is the position of the elevator in the inertial frame, and \mathbf{r} is the position of the particle with respect to the elevator. Since

$$\mathbf{F}^{\text{ext}} = m\ddot{\mathbf{r}}' = m\ddot{\mathbf{R}} + m\ddot{\mathbf{r}}$$

the equation of motion in the elevator frame is

$$m\ddot{\mathbf{r}} = \mathbf{F}^{\text{ext}} - m\ddot{\mathbf{R}}$$

That is, the mass times the acceleration in the elevator frame is equal to the sum of the external forces plus a “pseudo force,” $-m\ddot{\mathbf{R}}$. In particular, if the elevator is in free fall in a uniform gravitational field, then the acceleration (in the elevator frame) is

$$m\ddot{\mathbf{z}} = mg - mg = 0$$

and the particle does not accelerate with respect to the elevator.

2. Rotation

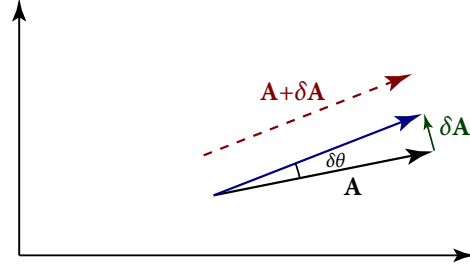
Rotation is a more interesting case. Consider a vector \mathbf{A} at rest in a frame that rotates *with respect to an inertial frame* through an infinitesimal angle $\delta\boldsymbol{\theta}$. As shown in Fig. 1, the change in \mathbf{A} is perpendicular to both \mathbf{A} and $\delta\boldsymbol{\theta}$ and has magnitude $\delta\theta A$. If \mathbf{A} had a component parallel to $\delta\boldsymbol{\theta}$, it would not be modified by the rotation; it is only the component in the plane perpendicular to the rotation axis that counts. The upshot is that the change in \mathbf{A} is given by

$$\delta\mathbf{A} = \delta\boldsymbol{\theta} \times \mathbf{A}$$

If the rotation takes place in time δt , then taking the limit as $\delta t \rightarrow 0$ we obtain

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\omega} \times \mathbf{A} \quad (4)$$

Figure 1: The frame in which \mathbf{A} is at rest rotates through angle $\delta\theta$, “displacing” \mathbf{A} to $\mathbf{A} + \delta\mathbf{A}$ (red). Translating \mathbf{A} back to align the tail of the vector to the original vector \mathbf{A} , allows us to identify $\delta\mathbf{A}$, the change in \mathbf{A} (green). Note that the vector direction of $\delta\theta$ is perpendicular to the plane of the figure and determined by the right-hand rule.



If, in addition, the vector \mathbf{A} itself is changing in the rotating frame, then the change in \mathbf{A} in the inertial frame is

$$\left(\frac{d\mathbf{A}}{dt} \right)_{\text{in}} = \left(\frac{d\mathbf{A}}{dt} \right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{A} \quad (5)$$

To understand Newtonian dynamics in the rotating frame, we will need to compute the acceleration, which requires us to apply Eq. (5) twice: first to get the velocity $\mathbf{v} = \dot{\mathbf{r}}$ and again to get the acceleration $\mathbf{a} = \ddot{\mathbf{r}}$:

$$\dot{\mathbf{r}}_{\text{in}} = \dot{\mathbf{r}}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r} \quad (6)$$

$$\ddot{\mathbf{r}}_{\text{in}} = \frac{d}{dt} (\dot{\mathbf{r}}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r})_{\text{rot}} + \boldsymbol{\omega} \times (\dot{\mathbf{r}}_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{r})$$

so that

$$\ddot{\mathbf{r}}_{\text{in}} = \ddot{\mathbf{r}}_{\text{rot}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \underbrace{2\boldsymbol{\omega} \times \dot{\mathbf{r}}_{\text{rot}}}_{\text{Coriolis}} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{centrifugal}} \quad (7)$$

Expressing Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}_{\text{in}}$, in terms of the quantities in the rotating frame, we get

$$m\ddot{\mathbf{r}}_{\text{rot}} = \mathbf{F} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - \underbrace{2m\boldsymbol{\omega} \times \dot{\mathbf{r}}_{\text{rot}}}_{\text{Coriolis}} - \underbrace{m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{centrifugal}} \quad (8)$$

where the three extra terms on the right are three **pseudo forces** that arise because we are using coordinates in the noninertial rotating frame. The first of these is usually negligible in situations of practical interest. The second is called the **Coriolis** force,¹ and the third is the **centrifugal** force. Let's consider each of them in turn.

To simplify things a bit, we will consider two-dimensional motions on a rotating turntable in a horizontal plane. The first pseudo force is nonzero during spin-up

¹Gaspard-Gustave de Coriolis (1792–1843) described this term in a paper of 1835, *Sur le principe des forces vives dans les mouvements relatifs des machines*, although Laplace had obtained the same term in connection with tides in 1778, while Giovanni Battista Riccioli and Francesco Maria Grimaldi had described it in connection with artillery in 1651. Evidently, Coriolis's name was first associated with this term early in the 20th century.

and points in the azimuthal direction. If you put a penny on a greased-up turntable at rest, start up the turntable in a counterclockwise direction seen from above, and watch the penny while standing on the axis of the turntable, you will see the penny move to the right (as the turntable slides beneath it) under the influence of this pseudo force.

Now stop the turntable, strip off the grease, and start the turntable up again. The third term (the centrifugal force) grows like ω^2 and points radially outward. Eventually, it exceeds the maximum static friction force the turntable can supply and the penny begins to slide radially outward.

At this point, the Coriolis force comes into play. The radial motion is outward, and the angular velocity vector is vertically up, so the Coriolis force acts to the right as viewed from the center of the turntable. Instead of moving straight outward under the centrifugal force, the penny veers to the right.

2.1 Alternative Derivation

I would like to offer an alternative approach to the derivation of the acceleration in the rotating frame, this time using the Lagrangian formalism. To that end, we need an expression for the kinetic energy, expressed in terms of our chosen generalized coordinates (which are Cartesian coordinates $\mathbf{r} = (x, y, z)$ in the rotating frame, measured with respect to an origin on the axis of rotation). In an infinitesimal time interval δt , this vector changes with respect to an inertial frame for two reasons:

1. \mathbf{r} changes: $\mathbf{r} \rightarrow \mathbf{r} + \delta\mathbf{r}$
2. the frame rotates: $\mathbf{r} \rightarrow \mathbf{r} + \delta\boldsymbol{\theta} \times \mathbf{r}$

Of course, if both changes happen at once, we have

$$\mathbf{r} \rightarrow \mathbf{r} + \delta\mathbf{r} + \delta\boldsymbol{\theta} \times \mathbf{r}$$

so that the rate of change in position (in the inertial frame) is

$$\mathbf{v} = \frac{\delta\mathbf{r} + \delta\boldsymbol{\theta} \times \mathbf{r}}{\delta t} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}$$

Since \mathbf{v} is measured with respect to an inertial frame, we may use it to compute the kinetic energy of the particle:

$$T = \frac{m}{2} [\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + 2\dot{\mathbf{r}} \cdot \boldsymbol{\omega} \times \mathbf{r} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})]$$

Note that each value of \mathbf{r} and $\dot{\mathbf{r}}$ refers to the generalized coordinates in the rotating frame, and I have colored each term to make it easier to track each term in the following.

Let us ignore any potential energy for the moment to focus on the kinetic energy in terms of the generalized coordinates $(x, y, z) = (r^1, r^2, r^3)$. We'll take it step by step. First, we need to express T in component form:

$$T = \frac{m}{2} [\dot{r}^i \dot{r}^i + 2\epsilon^{ijk} \dot{r}^i \omega^j r^k + (\epsilon^{ijk} \omega^j r^k)(\epsilon^{ilm} \omega^l r^m)]$$

Now we take the partial with respect to \dot{r}^n :

$$\frac{\partial T}{\partial \dot{r}^n} = \frac{m}{2} [2\dot{r}^n + 2\epsilon^{njk} \omega^j r^k]$$

The total time derivative of this quantity is

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{r}^n} \right) &= m [\ddot{r}^n + \epsilon^{njk} (\dot{\omega}^j r^k + \omega^j \dot{r}^k)] \\ &= m [\ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}]^n \end{aligned}$$

Taking instead the partial with respect to r^n gives

$$\frac{\partial T}{\partial r^n} = \frac{m}{2} [2\epsilon^{ijn} \dot{r}^i \omega^j + \epsilon^{ijn} \omega^j \epsilon^{ilm} \omega^l r^m + \epsilon^{ijk} \omega^j r^k \epsilon^{iln} \omega^l]$$

I would now like to show that the two terms in red in this last expression are the same. To manage this, I will use the permutation properties of the Levi-Civita symbols to bring the n to the front:

$$\begin{aligned} \frac{\partial T}{\partial r^n} &= \frac{m}{2} [2\epsilon^{ijn} \dot{r}^i \omega^j + -\epsilon^{nji} \omega^j \epsilon^{ilm} \omega^l r^m - \epsilon^{nli} \omega^l \epsilon^{ijk} \omega^j r^k] \\ &= m [-\epsilon^{nji} \dot{r}^i \omega^j - \{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\}^n] \\ &= -m [\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]^n \end{aligned}$$

Putting these results together, we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}^n} \right) - \frac{\partial L}{\partial r^n} = m [\ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})]^n + \frac{\partial U}{\partial r^n} = 0$$

In other words, the “Newtonian” equation of motion for \mathbf{r} is

$$m\ddot{\mathbf{r}} = -\nabla U - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Of course, this result is identical to Eq. (8). All terms involving \mathbf{r} are expressed in the rotating frame.

Exercise 2 Show that the angular deviation δ of a plumb line from the true vertical at a point on the Earth's surface at latitude λ (measured from the Equator) is

$$\delta = \frac{\frac{1}{2} R \omega^2 \sin 2\lambda}{g - R \omega^2 \cos^2 \lambda}$$

where R is the radius of the Earth. What is the greatest value (in arc-seconds) for this deviation?

Exercise 3 Compare the centrifugal acceleration at the Equator due to the Earth's rotation to the gravitational acceleration. By what factor is the centrifugal acceleration due to the Earth's rotation greater than the centrifugal acceleration due to its motion about the Sun. (Light takes about 500 seconds to travel from the Sun to the Earth.)

3. Foucault's pendulum

Jean Bernard Léon Foucault (1819–1868) made the first successful demonstration of the Earth's rotation using a pendulum at the Paris Observatory in February, 1851. Shortly thereafter, he suspended a 28-kg bob from a 67-m long wire from the dome of the Panthéon in Paris. The pendulum's plane of oscillation rotated clockwise 11° per hour. Why?

Let's assume that the pendulum is at latitude λ (measured with respect to the equator), as shown in Fig. 2. Then the Earth's angular velocity is

$$\boldsymbol{\Omega} = \Omega(\hat{\mathbf{z}} \sin \lambda - \hat{\mathbf{x}} \cos \lambda) \quad (9)$$

where $\hat{\mathbf{z}}$ points vertically up, $\hat{\mathbf{y}}$ points east, and $\hat{\mathbf{x}}$ points south. The Coriolis force is thus

$$\mathbf{F}_{\text{Coriolis}} = -2m\boldsymbol{\Omega} \times \dot{\mathbf{r}} = -2m\Omega[\hat{\mathbf{z}} \sin \lambda - \hat{\mathbf{x}} \cos \lambda] \times (\dot{x}\hat{\mathbf{x}} + \dot{y}\hat{\mathbf{y}} + \dot{z}\hat{\mathbf{z}}) \quad (10)$$

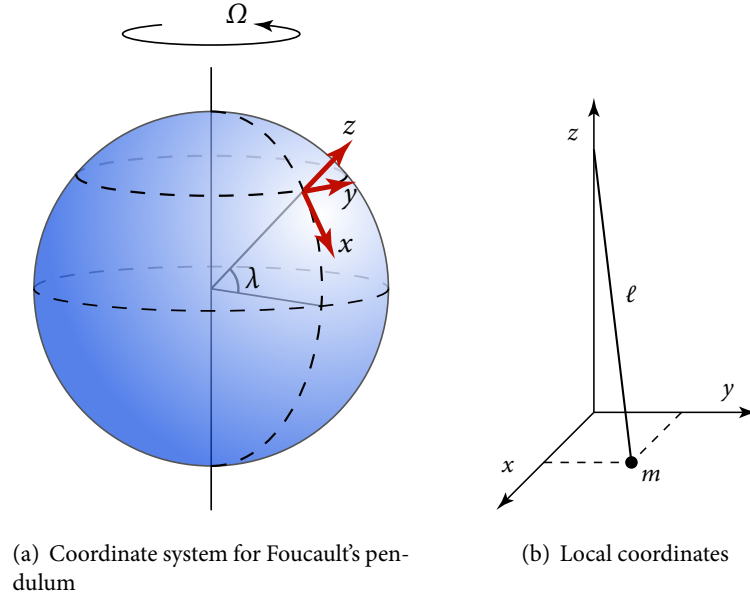


Figure 2: *Foucault's pendulum.*

For small-amplitude oscillations, \dot{z} is negligible. Then

$$\mathbf{F}_{\text{Coriolis}} \approx -2m\Omega [\sin \lambda (\dot{x}\hat{\mathbf{y}} - \dot{y}\hat{\mathbf{x}})] + \text{higher order terms} \quad (11)$$

Newton's equations in the xy plane then become

$$m(\ddot{x}\hat{\mathbf{x}} + \ddot{y}\hat{\mathbf{y}}) = -mg\left(\frac{x}{\ell}\hat{\mathbf{x}} + \frac{y}{\ell}\hat{\mathbf{y}}\right) - 2m\Omega \sin \lambda (\dot{x}\hat{\mathbf{y}} - \dot{y}\hat{\mathbf{x}}) \quad (12)$$

Let $\omega^2 = g/\ell$. Then Eq. (12) becomes the coupled equations

$$\ddot{x} + \omega^2 x - 2\Omega \sin \lambda \dot{y} = 0 \quad (13)$$

$$\ddot{y} + \omega^2 y + 2\Omega \sin \lambda \dot{x} = 0 \quad (14)$$

An elegant approach to solving these coupled equations is to pack x and y into the complex variable $z = x + iy$. Multiplying Eq. (14) by i and adding, we have

$$\ddot{x} + i\ddot{y} + \omega^2(x + iy) + 2\Omega_z(i\dot{x} - \dot{y}) = 0 \quad \implies \quad \ddot{z} + \omega^2 z + 2i\Omega_z \dot{z} = 0 \quad (15)$$

where $\Omega_z = \Omega \sin \lambda$ is the vertical component of the Earth's angular velocity. We now guess a solution of the form $z = z_0 e^{ist}$, substitute into Eq. (15), and convert the differential equation to a quadratic equation:

$$(-s^2 + \omega^2 + 2i\Omega_z s)z = 0 \quad \implies \quad s^2 + 2\Omega_z s - \omega^2 = 0 \quad (16)$$

with roots

$$s = -\Omega_z \pm \sqrt{\Omega_z^2 + \omega^2} \quad (17)$$

Of course, the oscillation frequency of the pendulum is many orders of magnitude greater than the rotation rate of the Earth, so to first order in Ω_z

$$s \approx -\Omega_z \pm \omega \left(1 + \frac{\Omega_z^2}{2\omega^2} \right) \approx -\Omega_z \pm \omega \quad (18)$$

The general solution for z is therefore

$$z = (Ae^{i\omega t} + Be^{-i\omega t})e^{-i\Omega_z t} \quad (19)$$

Suppose that at $t = 0$ the pendulum is aligned with the x direction. Then the term in parentheses is (and remains) real, since $A = B$ and the term is $2A \cos \omega t$. If we wait until $\Omega_z t = \pi/2$, then z will be purely imaginary and the pendulum oscillates in the y direction. Hence, the plane of oscillation rotates at angular frequency $\Omega_z = \Omega \sin \lambda$. At the latitude of Claremont, $34^\circ 5' 48''$ N, the plane would rotate with a period of 42.8 h. If we set up a Foucault pendulum with a circle of little pins surrounding it, why does this period imply that all the pins would be knocked over in 21.4 h?

Exercise 4 The latitude of the Panthéon in Paris is $48^\circ 50' 46''$ N. Calculate the rate of rotation of the plane of Foucault's pendulum and compare to that reported above.

Problem 5 – Dangerous Celebration? If a particle is projected vertically upward to a height h above a point on the Earth's surface at a northern latitude λ , show that it strikes the ground at a point

$$\frac{4}{3} \omega \cos \lambda \sqrt{8h^3/g}$$

to the west, if you neglect air resistance and assume h is small (compared to what?).

Problem 6 – Centrifugal Potential Consider a particle moving in a potential $U(r)$. Rewrite the Lagrangian in terms of a coordinate system in uniform rotation with respect to an inertial frame. Calculate the Hamiltonian and determine whether $H = E$. Is H a constant of the motion? If E is not a constant of motion, why isn't it? The expression for the Hamiltonian thus obtained is the standard formula $1/2mv^2 + U$ plus an additional term. Show that the extra term is the *centrifugal potential energy*.

Problem 7 Chemist Ferdinand Reich conducted early experiments on the Coriolis acceleration in 1831, shortly before Coriolis made his investigation in 1832–5. Reich dropped pellets down a 158.5-m-deep mine shaft in Freiberg, Germany, and observed a mean eastward deflection of 28.3 mm. The latitude of Freiberg is $\lambda = 51^\circ$.

- (a) Calculate the expected eastward deflection, clearly stating any simplifying assumptions you make.
- (b) Reich also found a small southerly deflection, as have subsequent investigators. This is a second-order Coriolis effect. The eastward velocity arises from the dominant vertical (z) motion; the eastward velocity (\dot{y}) produces a Coriolis acceleration in the x direction, proportional to ω^2 , where ω is the angular speed of the Earth. You also need to keep track of the variation of g and the centrifugal force with height, since they are also proportional to ω^2 . In fact, all terms are proportional to $\frac{h^2\omega^2}{g} \sin \lambda \cos \lambda$. According to Marion and Thornton (10-13), the total southerly deflection is

$$\frac{4h^2\omega^2 \sin \lambda \cos \lambda}{g}$$

But they're wrong. The correct leading coefficient is not 4. Find the correct coefficient analytically by keeping track of all terms through quadratic order in ω . Then check your work with an exact numerical calculation in *Mathematica*.