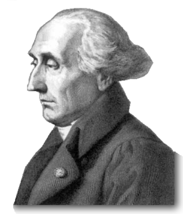
Accelerated Reference Frames, Lagrangian Approach

Monday, 28 November 2011

Physics 111



I would like to offer an alternative approach to the derivation of the acceleration in the rotating frame, this time using the Lagrangian formalism. To that end, we need an expression for the kinetic energy, expressed in terms of our chosen generalized coordinates (which are Cartesian coordinates $\mathbf{r} = (x, y, z)$ in the rotating frame, measured with respect to an origin on the axis of rotation). In an infinitesimal time interval δt, this vector changes with respect to an inertial frame for two reasons:

1. $\mathbf{r}$ changes: $\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}$
2. the frame rotates: $\mathbf{r} \to \mathbf{r} + \delta\boldsymbol{\theta} \times \mathbf{r}$

Of course, if both changes happen at once, we have

$$\mathbf{r} \to \mathbf{r} + \delta\mathbf{r} + \delta\boldsymbol{\theta} \times \mathbf{r}$$

so that the rate of change in position (in the inertial frame) is

$$\mathbf{v} = \frac{\delta\mathbf{r} + \delta\boldsymbol{\theta} \times \mathbf{r}}{\delta t} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}$$

Since $\mathbf{v}$ is measured with respect to an inertial frame, we may use it to compute the kinetic energy of the particle:

$$T = \frac{m}{2}\left[\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} + 2\dot{\mathbf{r}} \cdot \boldsymbol{\omega} \times \mathbf{r} + (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})\right]$$

Note that each value of $\mathbf{r}$ and $\dot{\mathbf{r}}$ refers to the generalized coordinates in the rotating frame, and I have colored each term to make it easier to track each term in the following.

Let us ignore any potential energy for the moment to focus on the kinetic energy in terms of the generalized coordinates $(x, y, z) = (r^1, r^2, r^3)$. We'll take it step by step. First, we need to express T in component form:

$$T = \frac{m}{2}\left[\dot{r}^i \dot{r}^i + 2\epsilon^{ijk} \dot{r}^i \omega^j r^k + (\epsilon^{ijk} \omega^j r^k)(\epsilon^{ilm} \omega^l r^m)\right]$$

Now we take the partial with respect to $\dot{r}^n$:

$$\frac{\partial T}{\partial \dot{r}^n} = \frac{m}{2}\left[2\dot{r}^n + 2\epsilon^{njk} \omega^j r^k\right]$$

The total time derivative of this quantity is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}^n}\right) = m\left[\ddot{r}^n + \epsilon^{njk}(\dot{\omega}^j r^k + \omega^j \dot{r}^k)\right]$$
$$= m\left[\ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times \dot{\mathbf{r}}\right]^n$$

Taking instead the partial with respect to r^n gives

$$\frac{\partial T}{\partial r^n} = \frac{m}{2}\left[2\epsilon^{ijn}\dot{r}^i\omega^j + \epsilon^{ijn}\omega^j\epsilon^{ilm}\omega^l r^m + \epsilon^{ijk}\omega^j r^k \epsilon^{iln}\omega^l\right]$$

I would now like to show that the two terms in red in this last expression are the same. To manage this, I will use the permutation properties of the Levi-Civita symbols to bring the n to the front:

$$\frac{\partial T}{\partial r^n} = \frac{m}{2}\left[2\epsilon^{ijn}\dot{r}^i\omega^j + -\epsilon^{nji}\omega^j\epsilon^{ilm}\omega^l r^m - \epsilon^{nli}\omega^l\epsilon^{ijk}\omega^j r^k\right]$$
$$= m\left[-\epsilon^{nji}\dot{r}^i\omega^j - \{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\}^n\right]$$
$$= -m\left[\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right]^n$$

Putting these results together, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}^n}\right) - \frac{\partial L}{\partial r^n} = m\left[\ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right]^n + \frac{\partial U}{\partial r^n} = 0$$

In other words, the "Newtonian" equation of motion for $\mathbf{r}$ is

$$\boxed{m\ddot{\mathbf{r}} = -\nabla U - m\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2m\boldsymbol{\omega} \times \dot{\mathbf{r}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}$$

Of course, this result is identical to the one described in the previous notes and in class. All terms involving $\mathbf{r}$ are expressed in the rotating frame.