Problem Set 7

Please staple problems 1+2 and 3+4 separately.

1. Distribution Function for Double Occupancy Statistics
   Let us imagine the new statistics in which the allowed occupancies of an orbital are 0, 1, and 2. The values of the energy associated with these occupancies are assumed to be 0, $\epsilon$, and $2\epsilon$, respectively.

   (a) Derive an expression for the ensemble average occupancy $\langle N \rangle$, when the system composed of this orbital is in thermal and diffusive contact with a reservoir at temperature $\tau$ and chemical potential $\mu$.

   (b) Return now to the usual quantum mechanics, and derive an expression for the ensemble average occupancy of an energy level which is doubly degenerate; that is, two orbitals have the identical energy $\epsilon$. If both orbitals are occupied, the total energy is $2\epsilon$. Compare your results for parts (a) and (b).

2. Fermi Gas in Two Dimensions (Schroeder 7.28)
   Consider a Fermi gas in two dimensions, confined to a square area $A = L^2$.

   (a) Find the Fermi energy (in terms of $N$ and $A$), and show that the average energy of the particles is $\epsilon_F/2$.

   (b) Derive a formula for the density of states, $g(\epsilon)$. You should find that it is a constant, independent of $\epsilon$.

   (c) Explain how the chemical potential of this system should behave as a function of temperature, both when $\tau \ll \epsilon_F$ and when $\tau \gg \epsilon_F$.

   (d) Because the density of states is a constant for this system, it is possible to find the chemical potential as a function of $N$ analytically. Do so, and show that the resulting expression has the expected qualitative behavior.

3. Symmetry of the Fermi-Dirac Distribution function (Schroeder 7.12)
   Consider two single-particle states, $A$ and $B$, in a system of fermions, where $\epsilon_A = \mu - x$ and $\epsilon_B = \mu + x$; that is level $A$ lies below $\mu$ by the same amount that level $B$ lies above $\mu$. Prove that the probability of level $B$ being occupied is the same as the probability of level $A$ being unoccupied. In other words, the Fermi-Dirac distribution function is “symmetrical” about the point where $\epsilon = \mu$. 

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4. **Paramagnetic Fermi Gas** (based on Schroeder 7.36)

Most spin-1/2 fermions, including electrons and $^3$He atoms, have non-zero magnetic moments. A gas of such particles is therefore paramagnetic. Consider, for example, a gas of free electrons, confined inside a three-dimensional box. The $z$ component of the magnetic moment of each electron is $\pm \mu_B$. In the presence of magnetic field $B$ pointing in the $z$ direction, each “up” state acquires an additional energy of $-\mu_B B = -\delta$, while each “down” state acquires an additional energy of $+\mu_B B = \delta$. (Do not confuse $\mu_B$, magnetic moment, with the chemical potential $\mu$!)

**(a)** Without doing any calculations, explain why you would expect the magnetization of a degenerate electron gas to be substantially lower than that of an ordinary paramagnet (discussed in Chapters 3.3 and 6.2 of Schroeder), for the same number of particles and field strength. Recall, that the magnetization is given by $\mu_B (N_\uparrow - N_\downarrow)$, where $N_\uparrow$ and $N_\downarrow$ are the numbers of electrons with up and down magnetic moments, respectively.

**(b)** Write down a formula for the density of states of this system in a presence of a magnetic field $B$, and interpret your formula graphically. (*Hint:* Think of the two spin orientations as two different types of particles and write down the density of states for each type. Keep in mind that the two types of particles are in diffusive equilibrium and therefore much have the same $\mu$, i.e. $\epsilon_F$.)

**(c)** Find a formula for the magnetization of this system at $\tau = 0$, in terms of $N$, $\mu_B$, $B$, and the Fermi energy. Assume that $\delta \ll \epsilon_F$. Compare your result to Eq. 3.35 in Schroeder, which describes magnetization for an ordinary paramagnet in the high-temperature limit.