Astronomy 62 Ann Esin
Introduction to Astrophysics 16 Mar 2012

Problem Set 8

Due: 23 Mar 2012

1. **Molecular Emission**
The rotational kinetic energy of a molecule is given by

$$E_{\rm rot} = \frac{1}{2}I\omega^2 = \frac{L^2}{2I}, \tag{1}$$

where L is the molecule's angular momentum and I its moment of inertia. The angular momentum is restricted by quantum mechanics to the discrete values

$$L = \hbar\sqrt{l(l+1)}, \tag{2}$$

where $l = 0, 1, 2, ...$

(a) For a diatomic molecule,

$$I = m_1 r_1^2 + m_2 r_2^2, \tag{3}$$

where m_1 and m_2 are the masses of the individual atoms and r_1 and r_2 are their respective separations from the center of mass of the molecule. Show that I can be written as

$$I = \frac{m_1 m_2}{m_1 + m_2} r^2, \tag{4}$$

where $r = r_1 + r_2$ is the total separation between atoms in the molecule, and the first factor is the reduced mass. (Recall our discussion of the reduced mass in the beginning of the semester.)

(b) The separation between the C and O atoms in CO molecule is approximately 1.2 Å, and the atomic masses of ^{12}C, ^{13}C and ^{16}O are 12.000 u, 13,003 u and 15.995 u, respectively ($1u = 1.66 \times 10^{-27}$ kg). Calculate the moments of inertia for ^{12}CO and ^{13}CO.

(c) What is the wavelength of a photon that is emitted by ^{12}CO during a transition between the rotational angular momentum states $l = 3$ and $l = 4$? Repeat you calculation for ^{13}CO. This is one method astronomers use to distinguish between different isotopes in the ISM.

2. **Dust Shrouding of Stars**

(a) Consider an F0V star entirely hidden inside a roughly spherical dust cloud of radius 100 AU. Describe the spectrum of the emission you will see from this object, i.e. what will be its total luminosity and the peak emission wavelength. You will find Appendix G useful. (You can assume for the purposes of this problem that the dust particles in the cloud are good blackbody emitters at all frequencies. This is not really true, but will not give a wildly wrong answer here.)

(b) Now, let's assume that the cloud is partly transparent, and our F0 star is still visible in the optical, but has visual (V band) extinction of 5 magnitudes. Estimate the average number density of dust particles in the cloud if the extinction is all due to dust particles with radii $a = 0.2\,\mu\text{m}$ and extinction coefficient in the V band $Q_V \sim 1.3$. The extinction coefficient is defined as $\sigma_V = Q_V \pi a^2$.

(c) The extiction coefficient depends on wavelength as $Q_\lambda \propto 1/\lambda$. What is the extinction in the B band if the extinction in the V band is 5 magnitudes? Based on your result, can you come up with a way to estimate extinction for any star with a known spectral type (which of course can be determined through spectroscopy)?

3. **Disk Formation in Protostars**

The radial acceleration of a proto-stellar cloud rotating with angular velocity ω is given by equation

$$\frac{d^2r}{dt^2} = -\frac{GM_r}{r^2} + r\omega^2. \tag{5}$$

When the total acceleration is equal to 0, we get the familiar result that the gravitational force is equal to the centripetal acceleration ($r\omega^2$).

(a) Use Eq. (5) and the conservation of angular momentum to show that the collapse of a cloud will stop (i.e. radial velocity, v_r, is equal to 0) in the plane perpendicular to its axis of rotation when the radius reaches

$$r_f = \frac{\omega_0^2 r_0^4}{2GM_r}, \tag{6}$$

where M_r is the interior mass, and ω_0 and r_0 are the original angular velocity and radius of the surface of the cloud respectively. Assume that the initial radial velocity of the cloud is zero and that $r_f \ll r_0$. You may also assume (incorrectly) that the cloud rotates as a rigid body during the entire collapse. *Hint:* Use the fact that

$$\frac{d^2r}{dt^2} = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}. \tag{7}$$

(Since no centripetal acceleration exists for collapse along the rotational axis, disk formation is a consequence of the original angular momentum of the cloud.)

(b) Assume that the original cloud had a mass of $1M_\odot$ and an initial radius of 0.5 pc. If collapse is halted at approximately 100 AU, find the initial angular velocity of the cloud.

(c) What was the original rotational velocity (in $\text{m}\,\text{s}^{-1}$) of the edge of the cloud?

(d) Assuming that the moment of inertia is approximately that of a uniform solid sphere, $I_{sphere} = \frac{2}{5}Mr^2$, when the collapse begins and a uniform disk, $I_{disk} = \frac{1}{2}Mr^2$, when it stops, determine the rotational velocity at 100 AU.

4. **Jeans Mass Revisited**

In class we derived the Jeans mass limit using the virial theorem. Another way to understand this condition for cloud collapse is to compare the free-fall time for the cloud with the sound crossing time. The free-fall time, t_{ff}, tells us how fast the cloud will shrink due to its own gravity if pressure is ignored. The sound crossing time, t_s, determines how fast the pressure can respond to counteract the pull of gravity.

(a) What is the correct condition for the cloud to initiate collapse, $t_{ff} < t_s$ or $t_{ff} > t_s$? Justify your answer.

(b) Consider a spherical isothermal cloud of radius R, with uniform density ρ_0, temperature T and gas pressure P. The sound crossing time is typically taken to be $t_s = R/v_s$, where v_s is the sound speed, defined so that $P = \gamma \rho_0 v_s^2$. Recall that γ is the adiabatic index, which we can set to 5/3 in a cloud consisting mostly of neutral atomic hydrogen. Use your result in part (a) to calculate the minimum radius necessary for the onset of the collapse as a function of ρ and T. The expression for free-fall timescale is derived in C&O, Eq. (12.26). (I gave a simplified derivation of t_{ff} in lecture, but you should use the more precise textbook result in your calculation.)

(c) Calculate the minimum mass necessary to initiate the collapse of our cloud. This is our new estimate of the Jeans mass. Compare your result to that derived in lecture and in the text, making sure that the functional dependencies on the cloud parameters (ρ and T) are the same.

(d) Is your minimum mass larger or smaller than the Jeans mass we derived in class? By what factor? Can you think of a reason for this discrepancy. (*Hint:* Think of the conditions under which the free-fall timescale was derived.)