Astronomy 62 | Ann Esin
Introduction to Astrophysics | 2 March 2012

Problem Set 7

Due: 9 Mar 2012

1. **Adiabatic Atmosphere and Greenhouse Effect**

   In HW#6 we constructed an isothermal (constant temperature) model of Earth's atmosphere. In reality, the atmospheric temperature decreases with height above the sea level, since it is mainly heated by absorption of the IR radiation from the Earth's surface. Since conduction and radiative heat transfer are very ineffective in our atmosphere, its temperature profile is set by convection.

   (a) Combine the equations for hydrostatic equilibrium and convective temperature gradient, to show that the temperature drop per height h in the atmosphere is given by

   $$\Delta T = T(R_\oplus) - T(R_\oplus + h) = \frac{\mu m_H}{k} g h \left(1 - \frac{1}{\gamma}\right), \quad (1)$$

   where $g = GM_\oplus/R_\oplus^2$ is gravitational acceleration on Earth's surface. (*Hint:* $h \ll R_\oplus$, so you can assume that the gravitational acceleration g is constant. The atmosphere also contains very little mass, so $M_r = M_\oplus$.) Evaluate your result for $h = 1\,\mathrm{km}$, keeping in mind that air contains mainly N_2, and its adiabatic exponent is roughly equal to $\gamma \simeq 1.4$.

   (b) The Earth's photospheric temperature is set by the thermal balance with the Sun; in HW#6 we calculated it to be $T_p = 254\,\mathrm{K}$. If the average temperature at sea level is about 280 K, calculate the height above sea level of the Earth's photosphere.

   (c) $C0_2$ is responsible for most of the atmospheric opacity to IR radiation. Increasing the amount of $C0_2$ in the atmosphere leads to an increase in the overall optical depth of the atmosphere and a displacement of the photosphere *upward* (i.e. farther from the Earth's surface). What happens to the surface temperature of the Earth under these circumstances? This result is generally known as the greenhouse effect.

2. **Nuclear Reactions**

   (a) Complete the following reaction sequences. Be sure to include any necessary leptons.

   $$\begin{aligned} {}^{27}_{14}\mathrm{Si} &\rightarrow {}^{?}_{13}\mathrm{Al} + e^+ + ? \\ {}^{?}_{13}\mathrm{Al} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{24}_{12}\mathrm{Mg} + {}^{4}_{?}\,? \\ {}^{35}_{17}\mathrm{Cl} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{36}_{18}\mathrm{Ar} + ? \end{aligned}$$

(b) Calculate the efficiency of energy generation from "burning" He through triple-α process. Repeat your calculations for C and O burning (for these two consider only lower-temperature reactions ${}^{12}_{6}\mathrm{C} +^{4}_{2}\mathrm{He} \rightarrow^{16}_{8}\mathrm{O}$ and ${}^{16}_{8}\mathrm{O} +^{4}_{2}\mathrm{He} \rightarrow^{20}_{10}\mathrm{Ne}$). Comment on how they compare with each other and with H burning. These processes will become important when we talk about stellar evolution. You might find the following masses useful,

$$\begin{aligned} m({}^{4}_{2}\mathrm{He}) &= 4.002603u; \\ m({}^{12}_{6}\mathrm{C}) &= 12.0u; \\ m({}^{16}_{8}\mathrm{O}) &= 15.99491u; \\ m({}^{20}_{10}\mathrm{Ne}) &= 19.99244u, \end{aligned}$$

where $u = 1.6605 \times 10^{-27}$ kg is the atomic mass unit, defined as 1/12 of the mass of ${}^{12}\mathrm{C}$.

3. **Polytropes**

We can solve the stellar structure equations analytically if we make an assumption that the pressure equation of state can be written in the form $P = K\rho^{(n+1)/n}$. Here K is a constant and n is called the polytropic index. Stellar models computed using this equation are called polytropes of index n; models with $n = 1.5$ (which produces the adiabatic equation $P = K\rho^{\gamma}$) and $n = 3$ describe quite well purely convective and purely radiative stars, respectively. Only models corresponding to $n = 0$, 1, 5 have analytic solutions. In this problem we will build a model with $n = 1$, i.e. $P = K\rho^2$.

(a) Combine this simplified equation of state with the equations of hydrostatic equilibrium and mass conservation to derive the following second-order differential equation for $\rho(r)$:

$$\frac{1}{x^2}\frac{d}{dx}\left[\frac{d\rho}{dx}x^2\right] = -\rho, \qquad (2)$$

where $x = r/r_0$ and $r_0^2 = 2K/(4\pi G)$.

(b) Verify that $\rho(x) = \rho_c \sin x/x$ is a solution to Equation (2). Sketch this function and find an expression for the radius of our model star, R. (*Hint:* Think of the boundary condition for ρ at the stellar surface.) What parameters does R depend on?

(c) In general, K is a known constant (it depends only on the physics of the equation of state). Show that the central density, ρ_c, in our model depends only on the total mass of the star, M. (*Hint:* Integrate ρ to find M.) Argue that this implies that $\rho(r)$, $P(r)$ and $T(r)$ are completely determined by M as well.

4. **Eddington Limit and Maximum Stellar Mass**

In this problem you will derive the fundamental physical limit on the luminosity of any object held together by gravity. Consider a balance of forces on a proton-electron pair located a distance $r > R$ from the center of a star with mass M, radius R and luminosity L. The proton in pulled inward by the star's gravity, while the electron is pushed outward by the radiation pressure from the star. If the force due to radiation pressure overwhelms gravitational force, the pair will be blown away.

(a) Show that the momentum gain for an electron in one photon-electron scattering is approximately $\Delta p \simeq h\nu/c$, where ν is the photon frequency.

(b) The force due to repeated scattering is $f_{\rm rad} = \Delta p/\Delta t$, where Δt is the approximate time between scatterings. Show that

$$f_{\rm rad} = \frac{L}{4\pi r^2}\frac{\sigma_T}{c}, \tag{3}$$

where $\sigma_T = 6.65 \times 10^{-29}\,\mathrm{m}^2$ is the Thomson cross section for photon-electron scattering. (We are using lower-case f to avoid confusion between force and flux.)

(c) Consider a star composed of pure hydrogen, and use Eq. (3) to show that for this star to remain stable, we must have

$$L < \frac{4\pi G M m_p c}{\sigma_T}. \tag{4}$$

The quantity on the right-hand side is called the Eddington Luminosity, and is generally denoted by $L_{\rm Edd}$. Note that it is independent of the distance from the star. Stars massive enough to have $L > L_{\rm Edd}$ will simply blow themselves apart.

(d) Combine Eq. (4) with the mass-luminosity relation, $L/L_\odot = (M/M_\odot)^{3.5}$ to estimate the maximum allowed mass for a main-sequence star.

(e) The presence of metals in atmospheres of real stars increases the cross section for electron scattering far above σ_T. How would this affect your result in part (d)?