Astronomy 62 | Ann Esin
Introduction to Astrophysics | 10 Feb 2012

Problem Set 4

Due: 17 Feb 2012

1. **Generalized Bohr Atom** (based on C&O 5.8)

   A hydrogenic ion is an atom with Z protons in the nucleus, and with all but one of its electrons lost due to ionization.

   (a) Starting with Coulomb's law, determine expressions for the orbital radii and energies for a Bohr model of such hydrogenic atom with Z protons.

   (b) Find the radius of the ground-state orbit, the ground-state energy and the ionization energy of singly ionized helium (He II).

   (c) Is there a difference in the energy levels of different isotopes of the same element, say $^3_2$He II and $^4_2$He II? *Hint:* the mass of an electron should really be replaced in your calculations by the reduced mass of the system, just as we did for the two-body problem.

2. **Hydrogen in a Box** (C&O, 8.9)

   Consider a gas of electrically neutral hydrogen gas that is maintained at a constant volume V. In this simple situation, the number of free electrons must equal the number of H II ions: $n_e V = N_{\rm II}$. Also, the total number of hydrogen atoms (both neutral and ionized), $N_{\rm tot}$, is related to the density of the gas by $N_{\rm tot} = \rho V/(m_p + m_e) \simeq \rho V/m_p$. where m_p is the mass of the proton. (The tiny mass of the electron can be safely ignored in the expression for $N_{\rm tot}$.) Let the density of the gas be $10^{-6}\,{\rm kg\,m^{-3}}$, typical of the atmosphere of an A0 star.

   (a) Derive the quadratic equation for the fraction of ionized atoms:

$$\left(\frac{N_{\rm II}}{N_{\rm tot}}\right)^2 + a\left(\frac{N_{\rm II}}{N_{\rm tot}}\right) - a = 0, \tag{1}$$

   where

$$a = \left(\frac{m_p}{\rho}\right)\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_{\rm I}/kT}. \tag{2}$$

   Hint: Start with the Saha equation.

   (b) Solve the equation in part (a) for the fraction of ionized hydrogen, $N_{\rm II}/N_{\rm tot}$. Numerically evaluate and plot your result for a range of temperatures between 5000 K and 25,000 K. Compare your graph with Fig. 8.8 in C&O.

3. **Ionization of He** (based on C&O, 8.10 and 8.11)

In this problem, you will follow a procedure similar to that of Example 8.1.4 for the case of a stellar atmosphere composed of pure helium to find the temperature at the middle of the He II partial ionization zone, where half of the He II atoms have been ionized. The ionization energies of neutral helium and singly ionized helium are $\chi_I = 24.6\,\text{eV}$ and $\chi_{II} = 54.4\,\text{eV}$, respectively. Use $n_e = 3 \times 10^{21}\,\text{m}^{-3}$ for the electron number density (since this ionization zone is found deeper inside the star, n_e is larger than the value we used in class).

(a) Explain why the relevant partition functions are $Z_I \simeq 1$, $Z_{II} \simeq 2$, and $Z_{III} = 1$.

(b) Find N_{II}/N_I and N_{III}/N_{II} for temperatures of $10,000\,\text{K}$, $40,000\,\text{K}$, and $60,000\,\text{K}$. Can you estimate the characteristic temperature of the He II partial ionization zone?

(c) Show that $N_{III}/N_{\text{total}} = N_{III}/(N_I + N_{II} + N_{III})$ can be expressed in terms of the ratios N_{II}/N_I and N_{III}/N_{II}.

(d) Make a graph of N_{III}/N_{total} similar to Fig 8.8 for a range of temperatures from $10,000\,\text{K}$ to $60,000\,\text{K}$. What is the temperature in the middle of the He II partial ionization zone? How does the answer compare to your estimate in part (b)? This layer is found fairly deep below stellar surface and plays a crucial role in pulsating stars.

4. **Giants vs. Dwarfs** (C&O 8.14)

Appendix G shows that giant stars, which have lower atmospheric densities, have slightly lower temperatures than the main-sequence stars of *same spectral type*. Use the Saha equation to explain why this is so. Note that this means that there is not a perfect correspondence between temperature and spectral type! Hint: Remember that the spectral type is determined by the relative strengths of absorption lines.