Astronomy 62 — Ann Esin
Introduction to Astrophysics — 14 Apr 2012

Problem Set 11

Due: 23 Apr 2012

Please staple problems 1+2 and 3+4.

1. **Redshifted Blackbody**

Consider a spherical blackbody of constant temperature and mass M, whose radius is R. An observer located at the surface of the sphere and a distant observer both measure the radiation given off by the sphere. Gravitational redshift preserves the characteristic shape of a blackbody spectrum, so both observers will see a blackbody spectrum.

(a) If the observer at the surface measures the luminosity of the blackbody to be $L(R)$, use the gravitational redshift and time dilation formulas, to show that the observer at infinity measures

$$L_\infty = L(R)\left(1 - \frac{2GM}{Rc^2}\right). \tag{1}$$

Hint: Think of the luminosity as number of photons emitted per unit time, times the average photon energy.

(b) Both observers use Wien's law, to determine the blackbody's temperature. Show that a distant observer will measure temperature T_∞ given by

$$T_\infty = T(R)\sqrt{1 - \frac{2GM}{Rc^2}}. \tag{2}$$

(c) Both observers use the Stefan-Boltzmann law to determine the radius of the spherical blackbody. Show that

$$R_\infty = \frac{R}{\sqrt{1 - 2GM/Rc^2}}. \tag{3}$$

Thus, using the Stefan-Boltzmann law without including the effects of general relativity will lead to an overestimate of the size of a compact blackbody.

(d) Calculate how much of an error one would make in inferring the radius of a neutron star from its emission properties, if General Relativistic effects are ignored.

2. **Accreting neutron stars**

Suppose that two stars are bound together in a binary system, and that one of these stars (the *secondary*) is losing mass to the other (the *primary*) at a rate $\dot{M} = dM/dt$. Take the radius of the primary star to be R_1 and its mass to be M_1.

(a) For simplicity, let's assume that all the energy released during accretion is lost through radiation from the surface of the primary (note that here we ignore the energy lost in the accretion disk). Assuming that the emerging radiation is a blackbody, show that the surface temperature of the primary star is

$$T_1 = \left(\frac{G M_1 \dot{M}}{4 \pi \sigma R_1^3} \right)^{1/4} . \tag{4}$$

Assume that the accreting material is distributed uniformly over the surface of the primary and the internal thermonuclear energy production of the primary is negligible.

(b) Particularly interesting cases arise when the primary is a white dwarf or a neutron star. Calculate the efficiency of rest mass energy release through accretion, i.e. find η such that

$$L = \eta \dot{M} c^2. \tag{5}$$

Estimate η for a neutron star.

(c) The accretion luminosity must always be below the Eddington limit:

$$L_{\rm Edd} = \frac{4 \pi G M m_p c}{\sigma_T}. \tag{6}$$

Calculate the maximum mass accretion rate (in units of g/s) possible for a neutron star and its corresponding surface temperature. In what waveband (radio, IR, optical, UV, X-ray or γ-ray) should we observe these systems? How luminous (in solar units, $L_\odot$) are they?

(Note that according to this simplistic analysis, accreting black holes will not produce any emission, since they do not have a "surface". In reality, a lot of the emission will be produced on route to the primary, in the accretion disk, and the emission characteristics of accreting black holes and neutron stars are actually very similar.)

(d) X-ray bursters are accreting neutron stars that periodically undergo explosive nuclear burning of He accumulated on their surface. If a burster releases 10^{32} J every 5 hours, what is the mass accretion rate onto the neutron star? Assume that all the accreted H is continuously converted to He on the neutron star surface, and only He is burned during the burst.

3. **Black Hole Tides**

One way massive isolated black holes can feed is by ripping apart passing stars due to their strong tidal forces. In this problem you will explore this effect.

(a) Suppose we have a body of mass m and radius r located a distance a away from a black hole of mass M. To estimate the tidal force on the body produced by the proximity of the black hole let's treat it as two halves, each of mass $m/2$, separated by a distance r. The difference between the gravitational forces on the half closer to the black hole and the half farther from the black hole is called the tidal force. In the limit when $r \ll a$, show that this force is equal to

$$F_T \simeq \frac{G M m r}{a^3}. \tag{7}$$

(b) Find the analytical expression for the tidal force at the black hole horizon. Use your result to estimate the tidal force on a person passing through the horizon of a $10M_\odot$ black hole. Repeat your calculation for a $10^9M_\odot$ black hole. What are your conclusions?

(c) Typically, black holes catch stars rather than people. Find the distance from the black hole at which a star of mass m and radius r will be disrupted by tidal forces, assuming that the two halves of the star are held together by gravity. Express your answer in terms of the back hole mass and the density of the star. Use your result to check whether a $1M_\odot$ star will get disrupted before it plunges through the horizon of a $10^8M_\odot$ black hole. (If the disruption occurs inside the horizon, there will be no visible effect, since any emission produced during this event will not make it out of the black hole.)

4. **The Central Black Hole – Sgr A***

Assume that the mass distribution in the center of our Galaxy is a combination of the constant density Galactic bulge and the central black hole. Then the mass interior to radius r is simply

$$M_r = \rho_0 r^3 + M_{\rm BH}, \tag{8}$$

where ρ_0 is the mass density of the bulge and $M_{\rm BH} = 4 \times 10^6 M_\odot$ is the mass of the black hole.

(a) Show that the velocity of a star in a circular orbit around the Galactic center is given by

$$v = \left[G\left(\rho_0 r^2 + \frac{M_{\rm BH}}{r}\right)\right]^{1/2}. \tag{9}$$

Calculate ρ_0, in units of solar masses per cubic parsec, if the orbital velocity is 250 km/s at $r = 300$ pc. You can assume that is this distance, black hole influence is negligible.

(b) Plot v as a function of $\log_{10} r$ over the range $0.01\,\text{pc} < r < 1000\,\text{pc}$. Express v in km/s and r is parsecs. Use your graph to estimate the distance at which the black hole begins to affect the stellar motions.

(c) Use your result for #3(c) to estimate the distance from the black hole at which a $1M_\odot$ main-sequence star will get tidally disrupted. Compare the result to your answer in part (b) and to the Schwarzschild radius of the central black hole.