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Core Lab: What Makes Things Tick?
Homework 2
Sept. 24, 2013

Homework 1

A mass m is suspended from a spring of spring constant k in a cylinder of a viscous fluid. The fluid produces a drag force on the mass of the form $F_d = -b\frac{dx}{dt}$, where b is a positive constant. The top of the spring is attached to a hook that is made to oscillate vertically with drive amplitude D and angular frequency $\omega = 2\pi f$. Solve for the steady-state amplitude and phase of the motion of the mass as a function of the drive frequency f. (That is, the phase of the mass's motion compared to the phase of the drive.)

Plot the amplitude A and phase ϕ of the mass's motion against f for $m = 200\text{g}$, $k = 10\frac{\text{N}}{\text{m}}$, $b = 0.1\frac{\text{kg}}{\text{s}}$, and $D = 2\text{cm}$. Your solution should use the complex exponential approach.

The equation governing this system's motion, from Newton's Laws, is

$$ma = -kx + k\left(D\sin\omega t - x\right) - bv, \;\; 0 = -kx + k\left(D\sin\omega t - x\right) - b\dot{x} - m\ddot{x}$$

Because sine and cosine are basically the same function, I can substitute $\left(D\cos\left(\omega t - x - \frac{\pi}{2}\right)\right)$ for $\left(D\sin\omega t - x\right)$. Isolating $\ddot{x}$, I can rewrite the above equation as

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{2k}{m}x = \frac{k}{m}D\cos\left(\omega t - \frac{\pi}{2}\right)$$

To find the homogeneous solution to this equation, I'll temporarily ignore the driving and set the left side of the equation above equal to zero. This gives me

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{2k}{m}x = 0$$

The characteristic polynomial of this equation is

$$\lambda^2 + \frac{b}{m}\lambda + \frac{2k}{m} = 0$$

Solving this using the quadratic equation, I find that

$$\lambda = \frac{\frac{-b}{m} \pm \sqrt{\frac{b^2}{m^2} - \frac{8k}{m}}}{2} = \frac{\frac{-b}{m} \pm \frac{b}{m}\sqrt{-\frac{8km}{b^2}}}{2} = \frac{-b}{2m} \pm i\frac{b\sqrt{\frac{8km}{b^2}}}{2m},$$

$$\lambda_1 = \frac{-b}{2m} + i\frac{b\sqrt{\frac{8km}{b^2}}}{2m}, \;\; \lambda_2 = \frac{-b}{2m} - i\frac{b\sqrt{\frac{8km}{b^2}}}{2m}$$

The homogeneous solution to the differential equation must then be, for some constants c_1 and c_2,

$$x_h = c_1 e^{\lambda_1} + c_2 e^{\lambda_2} = c_1 e^{\frac{-b}{2m} + i\frac{b\sqrt{\frac{8km}{b^2}}}{2m}} + c_2 e^{\frac{-b}{2m} - i\frac{b\sqrt{\frac{8km}{b^2}}}{2m}}$$

Because I'm trying to find the steady-state amplitude and phase of the mass's motion and these terms go to zero at high values of t, I don't actually care about this solution.

Because the driving function takes the form

$$\frac{kD}{m}\cos\left(\omega t - \frac{\pi}{2}\right) = \text{Re}\left[\frac{kD}{m}e^{\omega t - \frac{\pi}{2}}\right]$$

I can expect the steady-state solution of this differential equation to take a similar form, $Ae^{\omega t - \frac{\pi}{2}} = Ae^{-\frac{\pi}{2}}e^{\omega t} = Be^{\omega t}$. Plugging this expected solution into my differential equation, I find that

$$-B\omega^2 e^{\omega t} + i\frac{b}{m}B\omega e^{\omega t} + \frac{2k}{m}Be^{\omega t} = \frac{kD}{m}e^{\omega t}$$

Now cancelling all the $e^{\omega t}$s, I find that

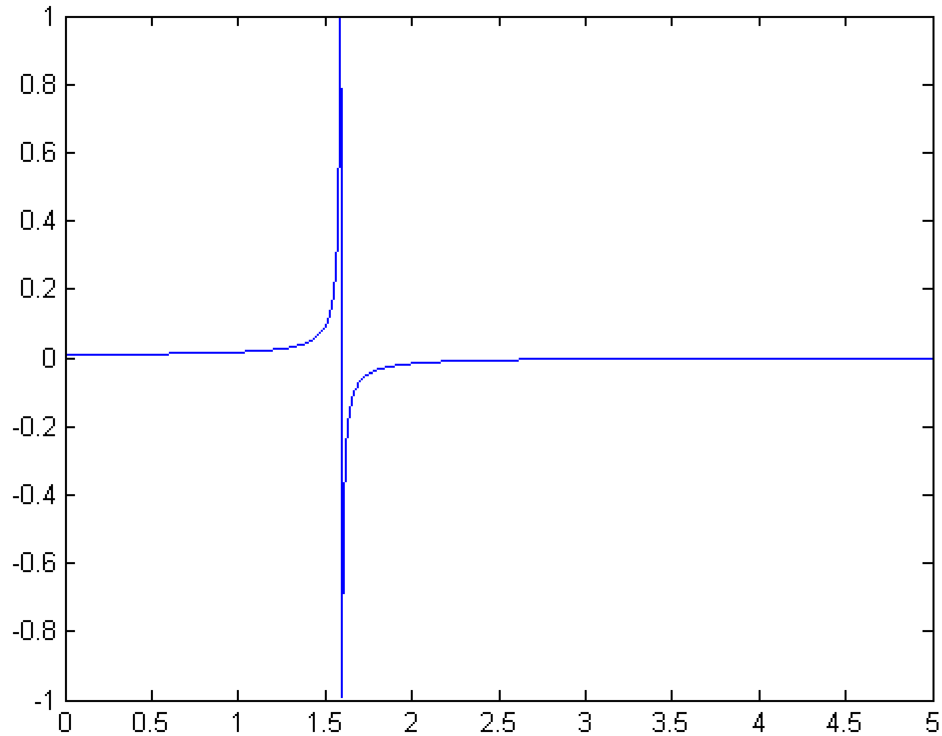
$$B\left(-\omega^2 + i\frac{b}{m} + \frac{2k}{m}\right) = \frac{kD}{m}, \quad B = \frac{kD}{m\left(-\omega^2 + i\frac{b}{m} + \frac{2k}{m}\right)} = \boxed{\frac{kD}{-4m\pi^2 f^2 + ib + 2k}}$$

That thing in the box is then the steady-state amplitude of the motion of the mass, and the phase is a constant $\boxed{\omega t}$.

Here's the plot of amplitude (after a long time), coming from the MATLAB expression

```
y = real((k.*D)./(-4.*m.*pi.*pi.*x.*x + j*b + 2.*k));
```

after I set up a vector called "x" that went from 0.001 to 5.000 by increments of 0.001, and filled in the proper variable values:



And here's the graph of phase (ω):