Characterizing the Frequency Response of a Damped, Forced Two-Mass Mechanical Oscillator

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Abstract

A two-mass oscillator was constructed using two carts, springs, and a damper on a track. A model for the amplitude and phase of each carts displacement for an input frequency was developed through theoretical analysis. The system was found to have two resonant frequencies and to exhibit a "phase-flip," where one of the carts would be opposite in phase to the other one for a range of frequencies, but flip so that it was in phase with the other at sufficiently low frequencies. Numerical data taken from the experiment failed to verify the model, as the data was likely processed incorrectly.

Introduction

Many systems can be modeled as a two-mass oscillator, notably atomic bonds. Such systems often have some damping present. Subject to a period forcing function, these systems will also display resonance at a certain frequency or frequencies. This experiment investigated the frequency response of a two-cart mechanical system, representing one possible configuration of dampers, masses, and springs.

Theory

The model for the experiment is represented in Fig. 1. In this particular two-mass oscillator, only the left mass, m_1, is damped.

Assume that the input displacement y is sinusoidal and can thus be written as $y = Ae^{i\omega t}$. Then assume that the steady-state responses of x_1 and x_2 will oscillate at the same frequency, so $x_1 = X_1 e^{i\omega t}$ and $x_2 = X_2 e^{i\omega t}$ [1]. By Newton's second law of motion, the governing system of equations [2] is

$$m_1\ddot{x}_1 = -k_1(x_1 - y) - k_2(x_1 - x_2) - c\dot{x}_1 \tag{1}$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1) - k_3 x_2 \tag{2}$$

For the purposes of this experiment, it is assumed that $k = k_1 = k_2 = k_3$ and $m = m_1 = m_2$. From the above assumptions, we can solve the differential equations for X_1 and X_2, combine the expressions into a single complex exponential, and obtain the amplitudes and phases of X_1 and X_2 (A_1, A_2 and ϕ_1, ϕ_2 respectively) as functions of the drive frequency ω.

$$A_1 = \frac{kA\left(2 - \frac{m}{k}\omega^2\right)}{\sqrt{\left(\frac{m^2}{k}\omega^4 - 4m\omega^2 + 3k\right)^2 + (cm\omega^3 - 2ck\omega)^2}} \tag{3}$$

$$A_2 = \frac{kA}{\sqrt{\left(\frac{m^2}{k}\omega^4 - 4m\omega^2 + 3k\right)^2 + (cm\omega^3 - 2ck\omega)^2}} \tag{4}$$

$$\phi_1 = \phi_2 = -\arctan\frac{cm\omega^3 - 2ck\omega}{\frac{m^2}{k}\omega^4 - 4m\omega^2 + 3k} \tag{5}$$

Experiment

The apparatus for the experiment is shown in Fig. 2. The model discussed in the previous section was recreated with two carts, three springs, and a magnetic damper on a track. Rather than measuring the linear displacement of the motor input, the angular displacement was measured and later converted into linear displacement.

The motion sensor was set in front of one of the carts so that it would detect a cardboard flag of negligible mass attached to the cart. DataStudio started data collection after the voltage source was switched on and the motor began to turn. Data collection continued until after the system apparently remained in steady state for a few seconds. Between each run, the system was allowed to return to rest and the voltage was adjusted. The system's responses to voltages of 10.5V to 1.5V (uncertainty of ±0.2V) were observed and recorded. After sufficient data had been collected for one cart, the motion sensor and flag were moved to the other cart. Raw data was recorded in the form of position and time in DataStudio. [insert figure of DataStudio display]

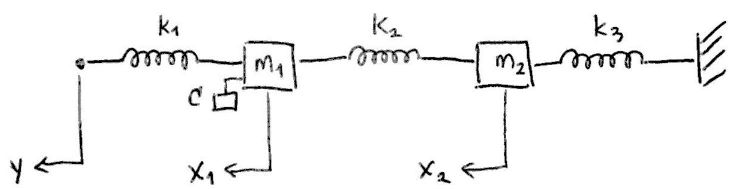

Figure 1: Representation of the system model. The input displacement y is applied to the leftmost spring. Mass m_1, attached to two springs with spring constants k_1 and k_2 on opposite sides, is damped by a factor of c and has horizontal displacement x_1. Mass m_2, attached to a spring of spring constant k_3 with a fixed end, has horizontal displacement x_2.

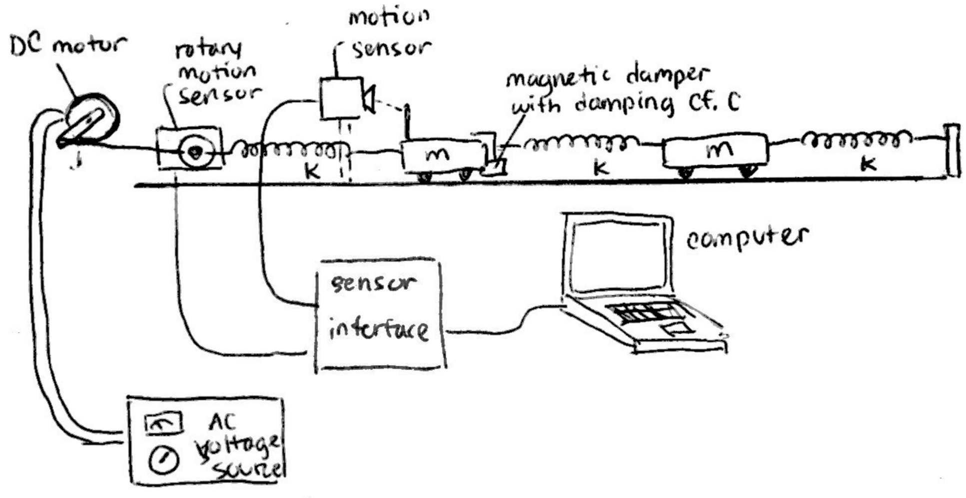

Figure 2: Apparatus of the oscillating system and instruments used to measure the system's motion. The DC motor provided a sinusoidal displacement with amplitude $A = 0.021 \pm 0.005$ m and frequency directly proportional to the input voltage. The two carts are each of mass $m = 0.600 \pm 0.0001$ kg and are attached to springs, each of spring constant $k = 10 \pm 0.1$ N/m. The damping coefficient of the magnet was not directly measured, but estimated to be $c = 0.02 \pm 0.01$ kg·m. A rotary motion sensor records the input displacement while a sonar motion sensor records one of the cart's displacement. Both sensors connect to an interface which sends data to DataStudio.

Results

The raw data from DataStudio was analyzed using an Igor procedure which called upon the program's sinusoidal curvefitting function to determine the amplitude, frequency, and phase of the steady state response. The resulting amplitudes and phases were both plotted against the driving frequencies that had been determined by the fit. Comparisons of these measurements to theoretical analysis are illustrated in Fig. 3.

Fitting the measured amplitude and phase data to the theoretical expressions yielded large $\tilde{\chi}^2$ values, greater than an order of magnitude of 2, for the amplitude A_1 and phases ϕ_1 and ϕ_2. For the amplitude A_1, the relatively sane value of $\tilde{\chi}^2 = 9.9$ was achieved only with extremely large uncertainties in the fit coefficients. While the predicted amplitude curves visually fit the data, the predicted phase curves do not seem to fit the data at all. Theoretical analysis predicts that both curves should be exactly equal. As such, none of the data could be considered good fits to their

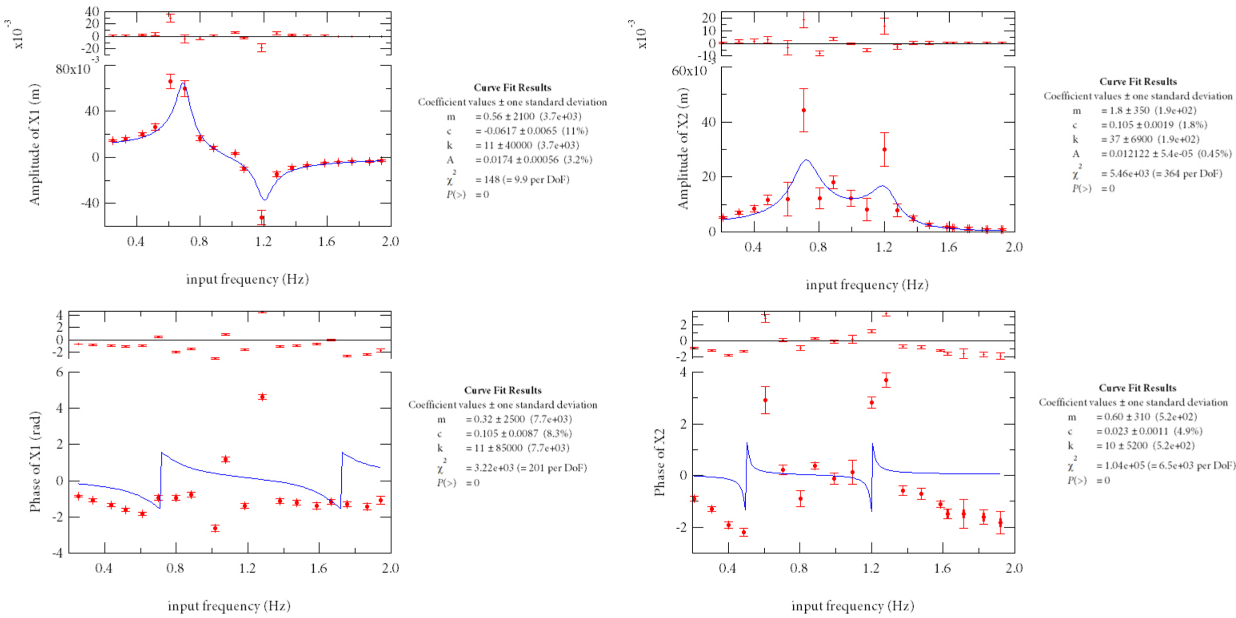

Figure 3: Amplitudes A_1, A_2 and phases ϕ_1, ϕ_2 as functions of frequency ω. Smooth curves are a fit to Eqs. 1, 2, 3, and 4 discussed in the theory section, yielding the fit coefficients shown in the figure with $\tilde{\chi}^2(A_1) = 9.9$, $\tilde{\chi}^2(A_2) = 364$, $\tilde{\chi}^2(\phi_1) = 202$, and $\tilde{\chi}^2(\phi_2) = 6.5 \times 10^3$. The upper panel shows residuals, which appear to be very large due to overly constrained uncertainties.

corresponding theoretical expressions.

However, both the data and theoretical analysis for the amplitude response of the system show that the system has two resonance frequencies, where the amplitude of a cart's steady state motion is at a maximum. These frequencies are approximately 0.7 Hz, where the undamped cart moves with maximum amplitude, and 1.2 Hz, where the damped cart moves with maximum amplitude This is a reasonable expectation, given that were the damped and undamped carts to be separated into smaller systems and subjected each to different frequencies, one would find that the two smaller systems have two distinct resonance frequencies. I suspect that the separation of the two peaks of the graph, the two resonance frequencies, is dependent upon the damping coefficient c.

A characteristic of the system that I cannot intuitively see from inspecting the phase graphs is a phase-flip as the system transitions from high to low frequencies. My initial observations of the system, before taking any data, saw that the two carts would be totally opposed in phase (one would be displaced to the right and the other displaced to the left, with the undamped cart in phase with the input) at high frequencies and at some threshold frequency, would "flip" to being in phase with each other. The phase-flip always seemed to occur in the damped cart. The undamped cart would always remain in phase with the input displacement. It is much easier to discern this two-mode behavior from the graph of A_1, as the amplitude flips from postive to negative value at approximately 0.9Hz.

Conclusion

In this experiment, I was able to observe some fundamental characteristics of the two-cart mechanical oscillator: 1) it has two resonance frequencies, one for each mass, and 2) that there are two apparent modes of the system, one where the carts are in phase, and one where the carts are opposite in phase. However, it was difficult to model this behavior both through theoretical analysis by physics fundamentals and through numerical data. My theoretical analysis required simplifying assumptions (e.g. all the spring constants were equal) that may have made the model inaccurate. The greatest source of error in my experiment and the main reason why my measurements did not match theoretical expectations was how I processed the raw data from DataStudio. During this experiment, I learned to program my first Igor Procedure, and the method by which the procedure calculated the phase difference between the input frequency and the cart's displacement was probably wrong. If I were to re-analyze the raw data with a better Igor procedure, I would likely find a better fit with the predicted phase curve.

References

[1] P. N. Saeta, *private communcation*, 06 November 2013.

[2] L. Orwin, *lecture notes*, ENGR059, Harvey Mudd College, 17 October 2013.