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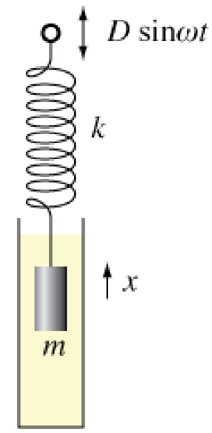
What Makes Things Tick
Homework 2
September 24th, 2013

A mass m is suspended from a spring of spring constant k in a cylinder of a viscous fluid. The fluid produces a drag force on the mass of the form $F_d = -b\,dx/dt$, where b is a positive constant. The top of the spring is attached to a hook that is made to oscillate vertically with drive amplitude D and angular frequency $\omega = 2\pi f$. Solve for the steady-state amplitude and phase of the motion of the mass as a function of the drive frequency f. (That is, the phase of the masss motion compared to the phase of the drive.)



Plot the amplitude A and phase ϕ of the mass's motion against f for $m = 200$g, $k = 10$N/m, $b = 0.1$kg/s, and $D = 2$cm. Your solution should use the complex exponential approach.

HINTS

- See the page on the theory of oscillations for a discussion of oscillatory systems, the final page of which illustrates the use of complex exponentials.
- Work symbolically; only use numerical values to produce the final plot.
- You can use Igor Pro to make the plot, if you wish.

The isolated forces on the mass give us the following equation:

$$\begin{aligned} F_{total} = ma &= -mg - bv - kx + drive \\ m\ddot{x} &= -mg - b\dot{x} - kx + kD\sin(2\pi f t) \\ m\ddot{x} + b\dot{x} + kx &= -mg + kD\sin(2\pi f t) \\ \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x &= -g + \frac{kD}{m}\sin(2\pi f t) \end{aligned}$$

This is now in the standard form for forced second order DEs, which we can try to solve using complex exponentials.

Homogenous Solution

We can guess that the answer to the homogeneous form has the form $x = e^{st}$:

$$\begin{aligned}
x &= e^{st} \\
0 &= \ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x \\
0 &= s^2 e^{st} + \frac{b}{m} s e^{st} + \frac{k}{m} e^{st} = s^2 + \frac{b}{m}s + \frac{k}{m} \\
s &= \frac{-\frac{b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{k}{m}}}{2} = \frac{-b \pm \sqrt{b^2 - 4km}}{2m}
\end{aligned}$$

This leaves us with the following homogeneous solution(s):

$$\begin{aligned}
x(t) &= c_1 x_1 + c_2 x_2 \\
&= c_1 e^{-\frac{b}{2m}t} e^{\frac{\sqrt{b^2-4km}}{2m}t} + c_2 e^{-\frac{b}{2m}t} e^{-\frac{\sqrt{b^2-4km}}{2m}t}
\end{aligned}$$

$$x_1 = e^{-\frac{b}{2m}t} e^{\frac{\sqrt{b^2-4km}}{2m}t} = \begin{cases} e^{\frac{-b+\sqrt{b^2-4km}}{2m}t}, & \text{for } b^2 > 4km \\ e^{-\frac{b}{2m}t}(\cos(\frac{\sqrt{4km-b^2}}{2m}t) + i\sin(\frac{\sqrt{4km-b^2}}{2m}t)), & \text{for } b^2 < 4km \\ e^{-\frac{b}{2m}t}, & \text{for } b^2 = 4km \end{cases}$$

$$x_2 = e^{-\frac{b}{2m}t} e^{-\frac{\sqrt{b^2-4km}}{2m}t} = \begin{cases} e^{\frac{-b-\sqrt{b^2-4km}}{2m}t}, & \text{for } b^2 > 4km \\ e^{-\frac{b}{2m}t}(\cos(\frac{\sqrt{4km-b^2}}{2m}t) - i\sin(\frac{\sqrt{4km-b^2}}{2m}t)), & \text{for } b^2 < 4km \\ e^{-\frac{b}{2m}t}, & \text{for } b^2 = 4km \end{cases}$$

However, with the values we are given, we know this to be the case where $b^2 < 4km$:

$$\begin{aligned}
\omega_0 = \frac{\sqrt{b^2 - 4km}}{2m} &= \frac{\sqrt{(0.1\frac{\text{kg}}{\text{s}})^2 - 4 \cdot 10\frac{\text{kg}}{\text{s}^2} \cdot 200\text{g}}}{2 \cdot 200\text{g}} \\
&= \frac{\sqrt{(0.1)^2 - 4 \cdot 10 \cdot 0.2}}{2 \cdot .02} \times \frac{1}{s} \\
&= \frac{\sqrt{-7.99}}{0.4} \times \frac{1}{s} \\
&= 7.067i \,\text{Hz}
\end{aligned}$$

We are only really interested in the real part of this solution,

$$x(t) = c_1 e^{-\frac{b}{2m}t} e^{\omega_0 t} + c_2 e^{-\frac{b}{2m}t} e^{-\omega_0 t} = e^{-\frac{b}{2m}t}(c_1 e^{\omega_0 t} + c_2 e^{-\omega_0 t})$$

so we can combine the c's into a single complex value. This does not loose any information, because the even and odd properties of the sine and cosine functions reduce the two exponentials to some complex combination of phases and sinusoids anyway.

$$x(t) = c e^{-\frac{b}{2m}t} e^{\omega_0 t}$$

Particular Solution

Now we can reintroduce the complete, forced DE and find a particular solution which accounts for the forcing term, $-g + \frac{kD}{m}\sin(2\pi f t)$. This is a sinusoidal with a constant, but that can just as easily be stated as $-g + \frac{kD}{m}\cos(2\pi f t - \pi/2)$ so the constant disappears after the first derivative and the sinusoid loops back, so we can try a particular solution of the form $x_p(t) = a + A \cdot Re(e^{i(\omega t - \pi/2)}) = a + A \cdot Re(e^{-i\pi/2}e^{i\omega t}) = Re(a + A \cdot e^{-i\pi/2}e^{i\omega t})$ for $\omega = 2\pi f$. We will define a $z(t) = x(t) + iy(t)$, such that $Re(z) = x$. We can then solve in terms of the full. complex exponentials.

$$\begin{aligned}
\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x &= -g + \frac{kD}{m}\sin(2\pi f t) \\
&= -g + \frac{kD}{m}\cos(2\pi f t - \pi/2) \\
Re\left(\ddot{z} + \frac{b}{m}\dot{z} + \frac{k}{m}z\right) &= Re\left(-g + \frac{kD}{m}e^{-i\pi/2}e^{i2\pi f t}\right) \\
\ddot{z} + \frac{b}{m}\dot{z} + \frac{k}{m}z &= -g + \frac{kD}{m}e^{-i\pi/2}e^{i2\pi f t} \\
-\omega^2 A e^{-i\pi/2}e^{i\omega t} + \frac{b}{m}ie^{-i\pi/2}\omega A e^{i\omega t} + \frac{k}{m}a + \frac{k}{m}Ae^{-i\pi/2}e^{i\omega t} &= -g + \frac{kD}{m}e^{-i\pi/2}e^{i\omega t}
\end{aligned}$$

We can equate the constant term:

$$\begin{aligned}
\frac{k}{m}a &= -g \\
a &= -\frac{mg}{k}
\end{aligned}$$

and simplify/factor out parts of the exponential terms:

$$\begin{aligned}
\left(-\omega^2 + \frac{b}{m}i\omega + \frac{k}{m}\right)Ae^{-i\pi/2}e^{i\omega t} &= \frac{kD}{m}e^{-i\pi/2}e^{i\omega t} \\
\left(-\omega^2 + \frac{b}{m}i\omega + \frac{k}{m}\right)A &= \frac{kD}{m} \\
A &= \frac{\frac{kD}{m}}{-\omega^2 + \frac{b}{m}i\omega + \frac{k}{m}} \\
&= \frac{kD}{-m\omega^2 + bi\omega + k}
\end{aligned}$$

These combine to the following z:

$$z(t) = -\frac{mg}{k} + \frac{kD}{-m\omega^2 + bi\omega + k}e^{-i\pi/2}e^{i\omega t}$$

Combined Solution

The total solution would then take the form of the real part of

$$x(t) = Re\left[ce^{-\frac{b}{2m}t}e^{\omega_0 t} - \frac{mg}{k} + \frac{kD}{-m\omega^2 + bi\omega + k}e^{-i\pi/2}e^{i\omega t}\right]$$

with

$$\omega_0 = \frac{\sqrt{b^2 - 4km}}{2m} = 7.067i\,\text{Hz}$$
$$\omega = 2\pi f$$

I'll rephrase the solution into trigonometric functions

$$x(t) = Re\left(ce^{-\frac{b}{2m}t}e^{\omega_0 t} - \frac{mg}{k} + kDe^{-i\pi/2}\left[\frac{1}{-m\omega^2 + bi\omega + k}e^{i\omega t}\right]\right)$$

$$= Re\left(ce^{-\frac{b}{2m}t}\left[\cos(-i\omega_0 t) + i\sin(-i\omega_0 t)\right] - \frac{mg}{k}\right)$$
$$+ Re\left(-kDi\left[\frac{k - m\omega^2 - bi\omega}{(k - m\omega^2)^2 + b^2\omega^2}\left(\cos(\omega t) + i\sin(\omega t)\right)\right]\right)$$

$$= e^{-\frac{b}{2m}t}\left(c_R\cos(-i\omega_0 t) - c_I\sin(-i\omega_0 t)\right) - \frac{mg}{k}$$
$$+ Re\left(kD\frac{m\omega^2 i - ki - b\omega}{(k - m\omega^2)^2 + b^2\omega^2}\left(\cos(\omega t) + i\sin(\omega t)\right)\right)$$

$$= e^{-\frac{b}{2m}t}\left(c_R\cos(-i\omega_0 t) - c_I\sin(-i\omega_0 t)\right) - \frac{mg}{k}$$
$$+ \frac{kD}{(k - m\omega^2)^2 + b^2\omega^2}Re\left(\left(m\omega^2 i - ki - b\omega\right)\left(\cos(\omega t) + i\sin(\omega t)\right)\right)$$

$$= e^{-\frac{b}{2m}t}\left(c_R\cos(-i\omega_0 t) - c_I\sin(-i\omega_0 t)\right) - \frac{mg}{k} + \frac{kD\left(-b\omega\cos(\omega t) + (k - m\omega^2)\sin(\omega t)\right)}{(k - m\omega^2)^2 + b^2\omega^2}$$

Our solution, then, in its complex and real form, is

$$\boxed{\begin{aligned} z &= ce^{-\frac{b}{2m}t}e^{\omega_0 t} - \frac{mg}{k} + \frac{kD}{-m\omega^2 + bi\omega + k}e^{-i\pi/2}e^{i\omega t} \\ x &= e^{-\frac{b}{2m}t}\left(c_R\cos(-i\omega_0 t) - c_I\sin(-i\omega_0 t)\right) - \frac{mg}{k} + \frac{kD\left(-b\omega\cos(\omega t) + (k - m\omega^2)\sin(\omega t)\right)}{(k - m\omega^2)^2 + b^2\omega^2} \end{aligned}}$$

SteadyState

For the steady-state solution, we can ignore the term that decays to 0, and the constant term is simply a vertical offset, which doesn't affect the amplitude of the oscillation, so we can focus on the ω-dependent term. For ease of calculation, I will use the complex form:

$$z = ce^{-\frac{b}{2m}t}e^{\omega_0 t} - \frac{mg}{k} + \frac{kD}{-m\omega^2 + bi\omega + k}e^{-i\pi/2}e^{i\omega t}$$

for steady state this simplifies to

$$\rightarrow \frac{kD}{-m\omega^2 + bi\omega + k}e^{-i\pi/2}e^{i\omega t}$$

We really only care about magnitude and phase, which come from the coefficient, not the unit-circle oscillation of magnitude 1, so this messed up complex number, which is made from both the i's in the denominator and the complex exponential in the numerator, can be cleaned up slightly and given the values from the problem. ($m = 200\text{g}, \quad k = 10\frac{\text{N}}{\text{m}}$, $b = 0.1\frac{\text{kg}}{\text{s}}, \quad D = 2\text{cm}, \text{ and } \omega = 2\pi f,$)

$$\begin{aligned}
\frac{kD}{-m\omega^2 + bi\omega + k}e^{-i\pi/2} &= \frac{kDi}{m\omega^2 - k - bi\omega} \\
&= \frac{kDi}{m\omega^2 - k - bi\omega} \cdot \frac{m\omega^2 - k + bi\omega}{m\omega^2 - k + bi\omega} \\
&= \frac{kD}{(m\omega^2 - k)^2 + b^2\omega^2}\Big((m\omega^2 - k)i - b\omega\Big)
\end{aligned}$$

That means the magnitude is : (I will get rid of units in the second step by converting everything to mks and canceling. The freqency must then be in $/s$)

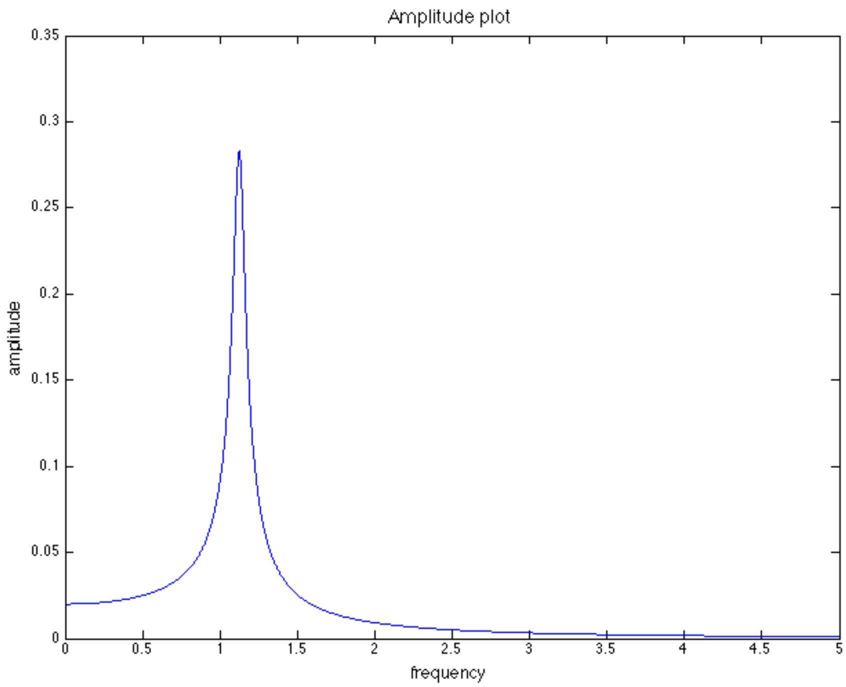
$$\begin{aligned}
A = \sqrt{Re^2 + Im^2} &= \frac{kD}{(m\omega^2 - k)^2 + b^2\omega^2}\sqrt{(-b\omega)^2 + (m\omega^2 - k)^2} \\
&= \frac{(10\frac{\text{N}}{\text{m}})(2\text{cm})\sqrt{\Big(-(0.1\frac{\text{kg}}{\text{s}})2\pi f\Big)^2 + \Big((200\text{g})4\pi^2 f^2 - (10\frac{\text{N}}{\text{m}})\Big)^2}}{((200\text{g})4\pi^2 f^2 - (10\frac{\text{N}}{\text{m}}))^2 + (0.1\frac{\text{kg}}{\text{s}})^2 4\pi^2 f^2} \\
&= \frac{0.2\sqrt{0.04\pi^2 f^2 + (0.8\pi^2 f^2 - 10)^2}}{(0.8\pi^2 f^2 - 10)^2 + 0.04\pi^2 f^2} \text{ in meters}
\end{aligned}$$

and the phase is

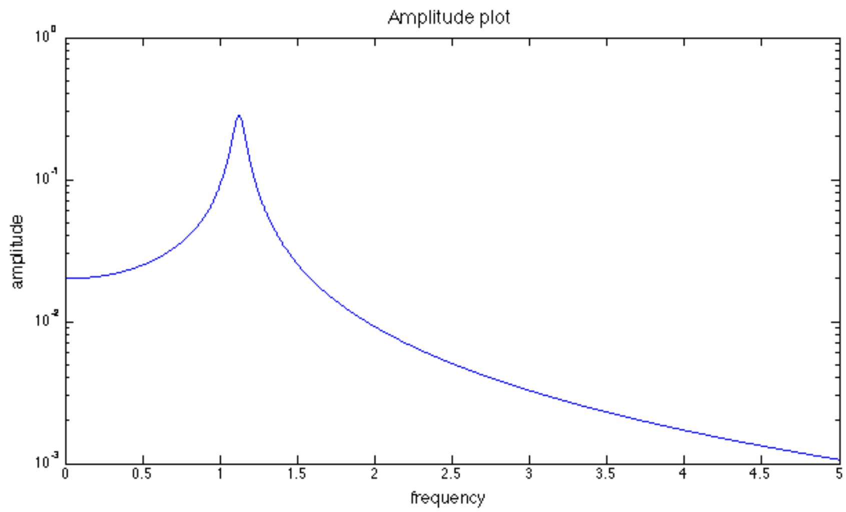
$$\begin{aligned}
\tan^{-1}\frac{Im}{Re} &= \tan^{-1}\frac{m\omega^2 - k}{-b\omega} \\
&= \tan^{-1}\frac{(200\text{g})4\pi^2 f^2 - (10\frac{\text{N}}{\text{m}})}{-(0.1\frac{\text{kg}}{\text{s}})2\pi f} \\
&= \tan^{-1}\frac{0.8\pi^2 f^2 - 10}{-0.2\pi f}
\end{aligned}$$

Here are plots of the amplitude and phase, with log and linear axis, for whatever preference.

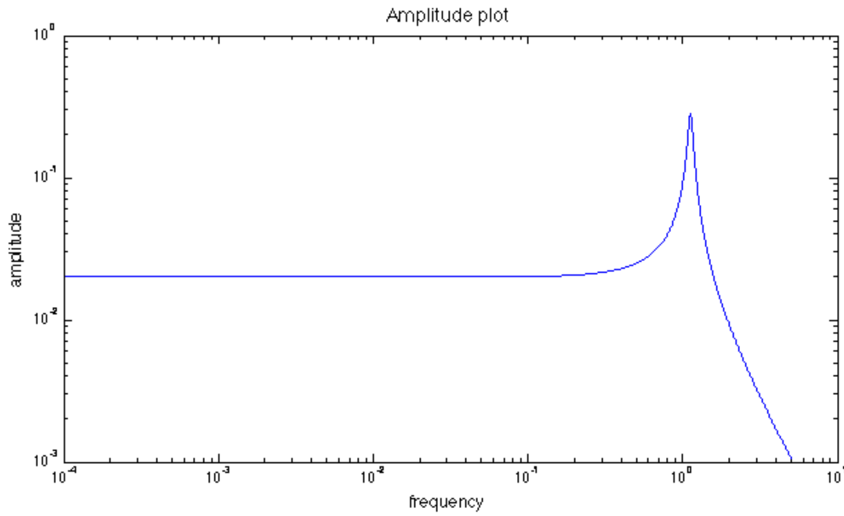
Linear



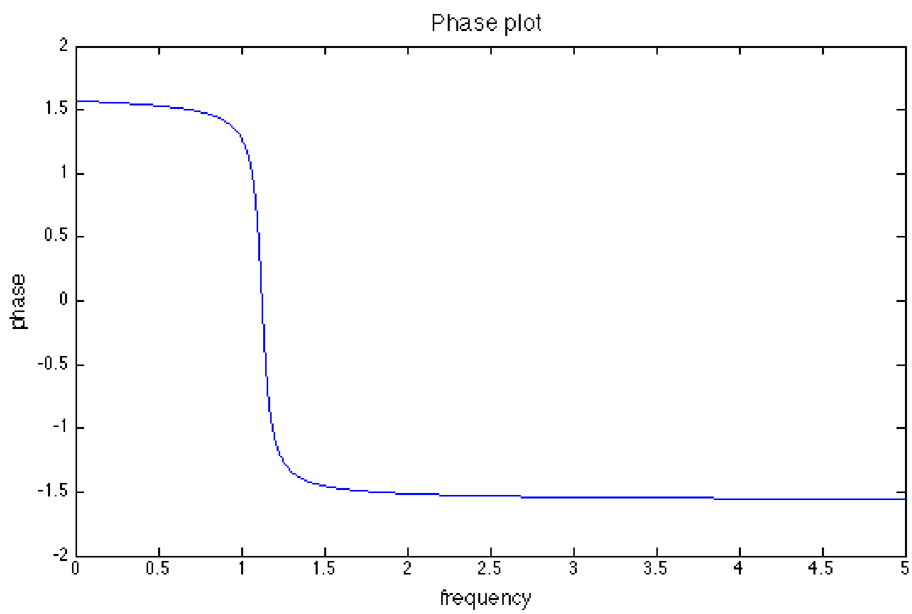
Single Log vertical axis



Log-log plot



And here is the phase:



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