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Core Lab 57 Section 01
Homework 02
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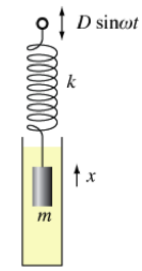
HW 02

A mass m is suspended from a spring of spring constant k in a cylinder of a viscous fluid. The fluid produces a drag force on the mass of the form $F_d = b\frac{dx}{dt}$, where b is a positive constant. The top of the spring is attached to a hook that is made to oscillate vertically with drive amplitude D and angular frequency $\omega = 2\pi f$. Solve for the steady-state amplitude and phase of the motion of the mass as a function of the drive frequency f. (That is, the phase of the masss motion compared to the phase of the drive.)



Plot the amplitude A and phase ϕ of the mass's motion against f for $m = 200$ g, $k = 10$ N/m, $b = 0.1$ kg/s, and $D = 2$ cm. Your solution should use the complex exponential approach.

The forces affecting the motion of the mass are the driving force $F_D = -Dk\sin\omega t$, the drag force $F_d = -b\frac{dx}{dt}$, and the force due to the spring $F_k = -kx$. The weight of the mass does not affect the motion of the mass because when we balance forces, it only affects the equilibrium position. Thus, when we sum our forces, we have

$$\begin{aligned}
F_{\text{total}} &= \sum F \\
ma &= F_d + F_k + F_D \\
m\ddot{x} &= -b\dot{x} - kx - Dk\sin\omega t \\
\ddot{x} &= -\frac{b}{m}\dot{x} - \frac{k}{m}x - \frac{k}{m}D\sin\omega t \\
\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x &= -\frac{k}{m}D\sin\omega t
\end{aligned}$$

Since we are looking for the steady state equation, going back to DE's, we are looking for the particular solution. Let $2\beta = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$. Also, we can say that $\sin\omega t$ is the the imaginary component of some complex exponential $e^{i\omega t}$. Let $x_P = Ae^{i\omega t}$ Therefore,

$$\begin{aligned}
\ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= -\frac{k}{m}De^{i\omega t} \\
-\omega^2 Ae^{i\omega t} + 2\beta i\omega Ae^{i\omega t} + \omega_0^2 Ae^{i\omega t} &= -\frac{k}{m}De^{i\omega t} \\
A\left(-\omega^2 + 2\beta i\omega + \omega_0^2\right) &= -\frac{k}{m}D \\
A &= \frac{k}{m}\frac{D}{\left(\omega^2 - 2\beta i\omega - \omega_0^2\right)}
\end{aligned}$$

We know that this A is our "phasor"; thus it follows the form $Xe^{i\theta}$. Using our STEMs knowledge,

$$Xe^{i\theta} = -\frac{k}{m}\frac{D}{\left(\omega^2 - 2\beta i\omega - \omega_0^2\right)}$$

Converting to polar form,

$$r = \sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(-2\beta\omega\right)^2} \qquad \theta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right)$$

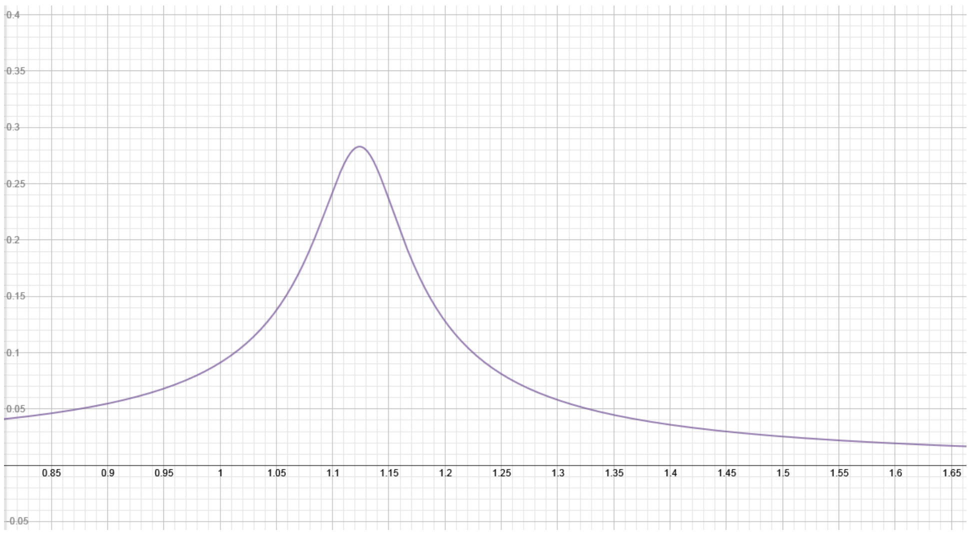
Therefore, our magnitude is

$$X = \frac{kD}{m\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(-2\beta\omega\right)^2}} = \boxed{\frac{kD}{m\sqrt{\left((2\pi f)^2 - \frac{k}{m}\right)^2 + \left(-\frac{b}{m}\left(2\pi f\right)\right)^2}}}$$

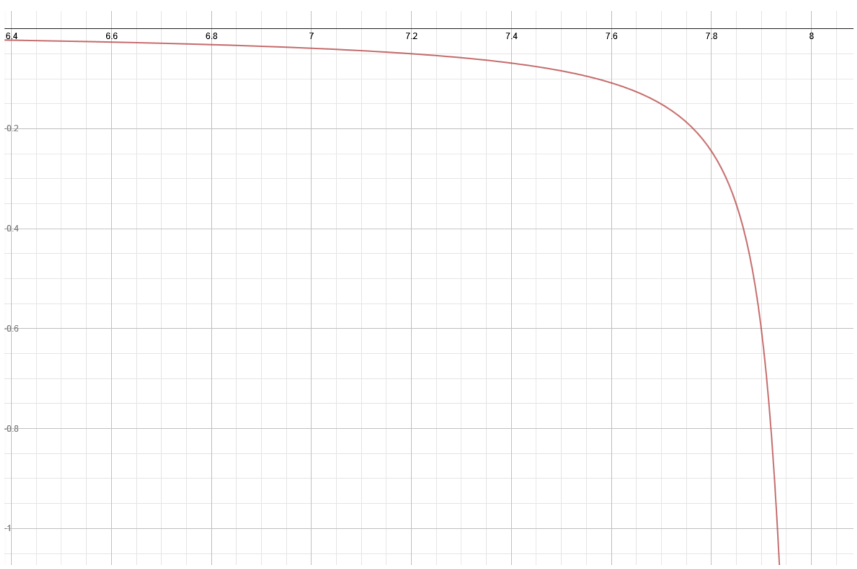
and our phase is

$$\phi = -\tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2}\right) = \boxed{-\tan^{-1}\left(\frac{\frac{b}{m}(2\pi f)}{\left(\frac{k}{m}\right)^2 - (2\pi f)^2}\right)}$$

Our plots are:
Amplitude verses Frequency



Phase versus Frequency



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