Interactions Between Two Non-Stationary Pendulums

Alexander Rich
Harvey Mudd College
3 December 2013

Abstract

Should two pendulums on a frictionless cart synchronize? This experiment measures the angular displacement of two pendulums on a cart. Because of minimal damping in the wheels of the cart and at the pivot points, the difference in angular displacement of the two pendulums follows a damped sinusoidal function, $20.4e^{-0.34t}\left[\cos(8t-2.4)+\sin(8t-2.4)\right]$. This fit resembled and closely followed the predicted behavior for the system with the measured parameters of mass of cart, length and mass of pendulum, and initial conditions.. Behavior was predicted using an Euler Lagrange approach.

1 Introduction

This experiment sought to uncover the perplexing nature of two pendulums on a cart. Inspiration came from the synchronization of metronomes on a freely moving plank. In this demonstration, several metronomes are placed on a plank that moves back and forth in the same plane that the metronomes swing. After a certain amount of time, the metronomes become synchronized.[1] Does the same phenomenon work with two pendulums? In this experiment, two pendulums are placed on a movable cart. One pendulum is given an initial angular displacement, then the system is allowed to oscillate freely.

2 Theory

The simple pendulum is a well understood system. It's behavior is sinusoidal, and if damped it will eventually stop swinging. The equation of motion can be written using $\sum\tau = I\ddot{\theta}$, where τ is the torque and I is the moment of inertia,

$$\ddot{\theta}+\frac{g}{l}\sin\theta = 0 \tag{1}$$

where g is the acceleration due to gravity and l is the length of the pendulum.

However, this assumes small angles and no unforced behavior. By putting the pendulums on a cart, there is now a forcing function on each pendulum, the torque axis is not constant, so the method used above is not feasible. Instead, the Lagrangian L is used:

$$L = T - u \tag{2}$$

where T is the kinetic energy and u the potential of the system.

For the multiple pendulum and cart system, the potential energy is simply gravitational and held by the pendulums:

$$u(t) = \ell mg(2 - \cos\theta - \cos\phi) \tag{3}$$

where m is the mass of a pendulum (For simplicity, the pendulums were the same mass), ℓ is the length of each pendulum, and θ and ϕ are the angular displacements from the vertical of each pendulum.

The kinetic energy is slightly more complicated, since the pendulums have both vertical and horizontal components. The kinetic energy is

$$T(t) = \frac{1}{2}\left(m_c\dot{x}^2 + m\left[(\dot{x} + \ell\dot{\theta}\cos\theta)^2 + (\ell\dot{\theta}\sin\theta)^2 + (\dot{x} + \ell\dot{\phi}\cos\phi)^2 + (\ell\dot{\phi}\sin\phi)^2\right]\right) \tag{4}$$

where m_c is the mass of the cart and x is the position of the cart.

Using Euler's equations, a function for x, θ, and ϕ can be found by solving the system

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} \tag{5}$$

and letting $q = x, \theta$, and ϕ.[2] Using Mathematica, it is possible to quickly determine the Euler equation for x, ϕ, and θ. In solving for x, a damping term is added (since most of the damping was attributed by the friction in the cart, so damping is not necessary for θ and ϕ). The equation solved was

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -\beta\dot{x} \tag{6}$$

where β is some damping coefficient, taken to be 0.34 kg/s as a reasonable damping term.

3 Experiment

The apparatus for the experiment is shown in Fig. 1. Two pendulums are hung from either side of a cart that rests on a track, where the pendulum's swing and the cart's track are in the same plane. A rotary motion sensor is placed on the pivot point of each pendulum, in order to record angular displacement as a function of time. The rotary motion sensor sends data to a Science Workshop 750, which then sends the data to DataStudio. From DataStudio, the data is importable to Igor Pro where it can be analyzed.

It was not feasible to measure the angular displacement of both pendulums in addition to the position of the cart (this would require an additional computer, since the Science Workshop 750 only allows for two such measuring devices), so the observed motion of the cart could not be compared with theoretical prediction.

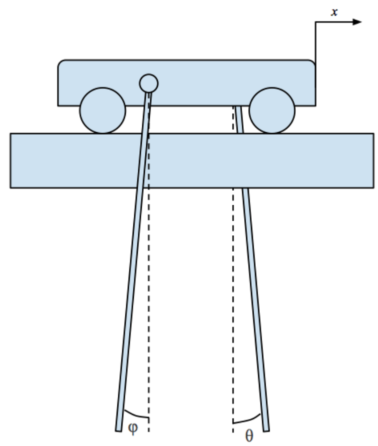

Figure 1: Apparatus to measure the relationship between the angular displacement of pendulum two (ϕ) and the angular displacement of pendulum one (θ) simultaneously. A rotary motion sensor was placed at the pivot of each pendulum to achieve this. The displacements were measured in opposite directions because, in the experiment, it became apparent that the two realized steady state where $\theta = \phi$. *Note:* the masses on each pendulum are not shown here.

Qualitatively, the cart oscillated first between $x = 0$ and a maximum then progressed to oscillating between a minimum and maximum centered about $x = 0$.

A mass (not shown in the figure) was also placed at the bottom of each rod. These masses ($m = 0.03$ kg) were equivalent for simplicity, the mass of the cart ($m_c = 0.15$ kg), was different. The heavier mass of the cart meant that the swinging pendulums were not able to influence the motion of the cart to the highest possible extent, but an effect was still obviously noticeable.

Initially, $\phi = 0$ (pendulum 2 was at rest) and θ was given a small displacement ($\theta_0 \approx \pi/16$). The two pendulums swung, and the recording stopped once the friction in the cart forced it to stop moving. At this point, the pendulums were swinging as if the pivots were stationary.

4 Results

Figure 2 shows a plot of θ as a function of time, both observed and predicted. Although there is some interesting behavior initially, after approximately five seconds, θ becomes much more predictable. The observed plot looks very similar to the predicted, the main difference being that it has a higher frequency and lower amplitude. This could be explained by inaccuracies in the measurement of system parameters, such as the length of the pendulum or the mass of the cart. Another source of deviation from the predicted behavior could be the lumping of the damping elements

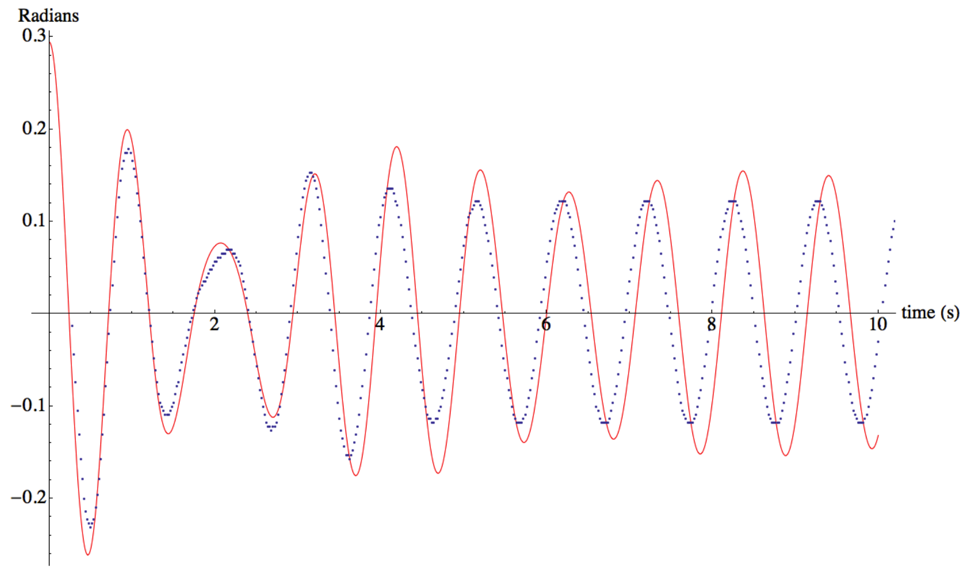

Figure 2: The graph displays the angular position θ as a function of time, both the observed and predicted plots. They look very similar, however it appears the period of the predicted plot is slightly longer than that of the actual observed plot. This could be a result of inaccurate measurements of the length or masses involved.

into the cart. In reality, damping occurs due to friction at the pivot point, because of air resistance about each pendulum, and from friction at the wheels of the cart. In the established theory, conversely, the only damping term included was on the wheels of the cart.

Although it was hypothesized that the two pendulums would fall into synchronization, in fact their steady-state behavior was exactly the opposite: they fell exactly out of phase with each other. Initially, their differences in angular displacement is high, however it sinusoidally converges to zero due to the damping of the cart. Figure 3 shows that the actual behavior of the pendulum closely matches that predicted by using the Lagrangian of the system.

One source of error, however, came from the cart being wide. Since each pendulum was on either side of the cart, they ended up being about 6 cm apart. When they were not precisely in phase, one pendulum could pull the cart on one side in one direction and the other could pull it in the other, so the two contributed to a net torque about the center of the cart. This torque caused the wheels of the cart to be stressed in unusual ways and to rub against the side of the track. Although the additional friction was likely not much, it was only present when the pendulums were at a certain position and thus not an easily reconcilable systematic error.

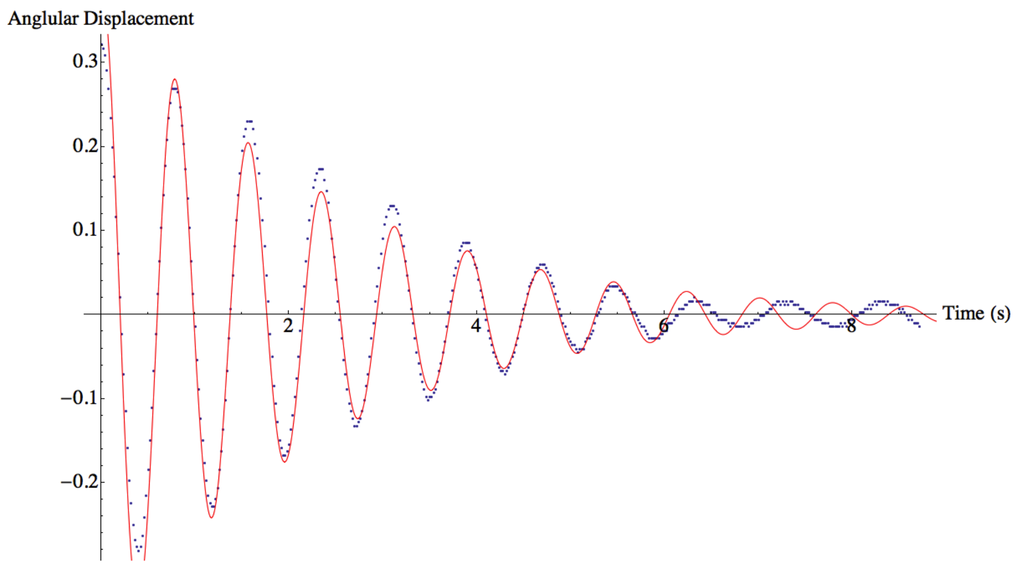

Figure 3: $\theta(t) - \phi(t)$. The observed points are recorded using the rotary motion sensor, and are adjusted for an offset. The theoretical plot comes from using the constants $l = 2.4, m = 0.05, m_c = 0.15, \beta = 0.34$. In a way, the determination of the damping term is cyclical, since it was chosen using the term from the fit function, however this is the most feasible method to get the system damping. Note that as time goes on, the observed data

5 Conclusion

Interestingly, although the pendulums were hypothesized to finally reach synchronization, as in the metronome experiment, they in fact reached exactly the opposite, ending up out of phase. While this behavior was both predicted and observed, it is important to gain an intuitive notion as to *why* the pendulums fell out of phase with each other.

Theoretically, if one were to remove the mass of the cart from the equations (but retain the damping due to friction at the wheels of the cart), the pendulums would be an underdamped system, however would eventually reach the same steady state of opposite phase.[3] If, on the other hand, one were to remove the damping but retain the mass of the cart, the pendulums will immediately be in a steady state behavior, and the difference between the phases will be a sinusoid with maximum value θ_0 and minimum value $-\theta_0$.

From these two hypothetical situations, it is apparent that the steady state

[3]This theoretical conclusion comes from setting various parameters to zero in the Mathematica code (e.g. mass or damping), affecting the solutions to Euler's equations. When these solutions are plotted, the resulting behavior can be predicted.

behavior in which the two pendulums oscillate out of phase is dependent on there being damping in the motion of the cart. In the metronome experiment, a plank was placed on top of two rollers, resulting in little to no damping.

More importantly, there were five metronomes used rather than two. Only based on the conclusion that pendulums will end up out of phase, an incorrect conclusion might be arrived at for a five pendulum system: It is impossible for five metronomes to be exactly out of phase. "Exactly out of phase" means the pendulums are π radians away from each other. But there are only two such positions on the unit circle with this characteristic. Thus only two pendulums can be exactly out of phase.

Perhaps the pendulums then achieve synchronization, as demonstrated by the metronome experiment. It would not make sense for the pendulums to be distributed equally about the entire 2π phase (e.g. 5 pendulums would be separated by $\frac{2\pi}{5}$ phase) because this would imply that, although each pendulum has two other pendulums at the equal spacing of phase, it also has pendulums at twice that ($\frac{4\pi}{5}$, $\frac{6\pi}{5}$, etc.).

It is likely that the most stable state for any number of pendulums greater than two is to be exactly in phase. An addition of another pendulum to the experiment could have tested this, or, since the theory for two pendulums matched the observation, adding a third pendulum term to the equations could also answer the question.

References

[1] Bahraminasab, Alireza. "Synchronisation." Online video clip. YouTube, 18 December 2007. Web. 21 November 2007. <http://youtu.be/W1TMZASCR-I>

[2] Weisstein, Eric W. "Euler-Lagrange Differential Equation." From MathWorld—A Wolfram Web Resource. <http://mathworld.wolfram.com/Euler-LagrangeDifferentialEquation.html>