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Core Lab: What Makes Things Tick?
Professor Saeta
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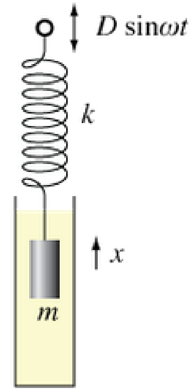
Homework 2

A mass m is suspended from a spring of spring constant k in a cylinder of a viscous fluid. The fluid produces a drag force on the mass of the form $F_d = -b\frac{dx}{dt}$, where b is a positive constant. The top of the spring is attached to a hook that is made to oscillate vertically with drive amplitude D and angular frequency $\omega = 2\pi f$. Solve for the steady-state amplitude and phase of the motion of the mass as a function of the drive frequency f. (That is, the phase of the masss motion compared to the phase of the drive.)



Plot the amplitude A and phase ϕ of the mass's motion against f for $m = 200$ g, $k = 10$ N/m, $b = 0.1$ kg/s, and $D = 2$ cm. Your solution should use the complex exponential approach.

Solution: The mass has several forces acting on it: force due to gravity, the drag force, and a force from the spring (which itself experiences a force from the drive). So, using the fact that $F = ma$,

$$m\frac{d^2x}{dt^2} = k(D\sin\omega t - x) - b\frac{dx}{dt} \tag{1}$$

Rewritten more physics-like:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{kD\sin\omega t}{m} \tag{2}$$

With complex exponentials, we can write the equation as

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{kD}{m}e^{i\omega t} \tag{3}$$

where $z = x + iy$, $\beta = \frac{b}{2m}$, and $\omega_0^2 = \frac{2k}{m}$. We are looking for a function $z(t) = x(t) + iy(t)$, so that we can take the real part of z and get x. Since we are only interested in the steady state solution, we only need the particular solution. The homogenous solution will end up being zero as time goes on, so we guess that the particular solution $z_p = Ae^{i\omega t}$. We have

$$\begin{aligned}
-\omega^2 Ae^{i\omega t} + 2\beta i\omega Ae^{i\omega t} + \omega_0^2 Ae^{i\omega t} &= \frac{kD}{m}e^{i\omega t} \\
A(\omega_0^2 - \omega^2 + 2\beta i\omega) &= \frac{kD}{m} \\
A &= \frac{kD}{m(\omega_0^2 - \omega^2 + 2\beta i\omega)}
\end{aligned}$$

To find the magnitude of A, we take the magnitude of the imaginary in the denominator:

$$A = \frac{kD}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2)}}$$

To plot the phase as a function of frequency, we use the following:

$$\phi = \tan^{-1}\left(\frac{2\beta\omega}{\omega_0^2 - \omega^2)^2}\right)$$

In all of these equations, the frequency is found by

$$\omega = 2\pi f.$$

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Problem discussed and/or checked with Mo Zhao.
