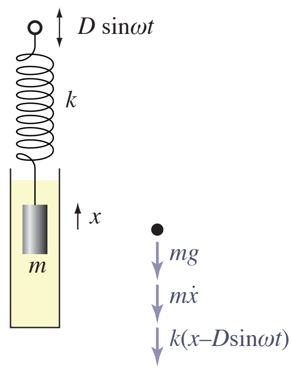
Homework Assignment 1, *What Makes Things Tick?*

Problem 1 A mass m is suspended from a spring of spring constant k in a cylinder of a viscous fluid. The fluid produces a drag force on the mass of the form $F_d = -b\,dx/dt$, where b is a positive constant. The top of the spring is attached to a hook that is made to oscillate vertically with amplitude D and angular frequency $\omega = 2\pi f$. Solve for the steady-state amplitude and phase of the motion of the mass as a function of the drive frequency f. (That is, the phase of the mass's motion compared to the phase of the drive.)

Plot the amplitude A and phase φ against f for $m = 200$ g, $k = 10$ N/m, $b = 0.1$ kg/s, and $D = 2$ cm. Your solution should use the complex exponential approach.

Solution: The equation of motion of the mass may be obtained by isolating the mass:



$$m\ddot{x} = -mg - b\dot{x} - k(x - D\sin\omega t)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{k}{m}D\sin\omega t - g$$

Define

$$\beta \equiv \frac{b}{2m}, \qquad \omega_0^2 = \frac{k}{m}, \qquad y = x - \frac{g}{\omega_0^2}$$

to give

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = D\omega_0^2 \sin\omega t \qquad (1)$$

where y measures the position of the mass *with respect to its equilibrium position*.

Equation (1) is an **inhomogeneous** differential equation. The full solution is a superposition (sum) of the solution to the *homogeneous* equation (in which we set the right-hand side to zero) and a *particular* solution. We will see that the homogeneous solution decays exponentially (it is proportional to $e^{-\beta t}$), so that in steady state, we will only need the particular solution. However, it's probably useful to go through this.

The homogeneous equation is

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = 0$$

We look for a complex exponential solution of the form $y = e^{i\Omega t}$, which we substitute into the homogeneous equation to give

$$-\Omega^2 y + 2\beta i\Omega y + \omega_0^2 y = 0 \implies (-\Omega^2 y + 2\beta i\Omega + \omega_0^2)y = 0$$

Either $y = 0$, which is the trivial solution, or the expression in the parentheses must vanish. It is a quadratic equation in the variable Ω, with solutions

$$\Omega_\pm = \beta i \pm \sqrt{-\beta^2 + \omega_0^2} = \beta i \pm \omega_1$$

where $\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$. Since this is a linear differential equation, we may superpose solutions, so the most general solution takes the form

$$y = A_+ e^{i\Omega_+ t} + A_- e^{-i\Omega_- t} = \left(A_+ e^{i\omega_1 t} + A_- e^{-i\omega_1 t}\right) e^{-\beta t}$$

As advertised, the homogeneous solution is proportional to $e^{-\beta t}$ and vanishes in the steady state.

Now we look for a particular solution to the inhomogeneous equation,

$$\ddot{y} + 2\beta\dot{y} + \omega_0^2 y = \omega_0^2 D \sin\omega t = \mathrm{Im}\left[\omega_0^2 D e^{i\omega t}\right]$$

where the last step follows from Euler's relation, $e^{i\theta} = \cos\theta + i\sin\theta$ and the fact that ω_0 and D are both real. We may look, therefore, for a complex solution that satisfies

$$y = \mathrm{Im}(z) = \mathrm{Im}\left(z_0 e^{i\omega t}\right)$$

Since the $\mathrm{Im}(a + b) = \mathrm{Im}(a) + \mathrm{Im}(b)$, the differential equation becomes

$$\mathrm{Im}\left[(i\omega)^2 z_0 e^{i\omega t} + 2\beta(i\omega) z_0 e^{i\omega t} + \omega_0^2 z_0 e^{i\omega t} = \omega_0^2 D e^{i\omega t}\right]$$

which will certainly be satisfied if the equality inside the brackets holds for the imaginary part and the real part of both sides. Factoring out $e^{i\omega t}$ gives

$$z_0 = D\frac{\omega_0^2}{\omega_0^2 - \omega^2 + 2\beta\omega i}$$

which is the complex amplitude of the mass's motion. To find its real amplitude and phase, we must simply express this number in polar form. I think the easiest way to do this is to work on the numerator and denominator separately (especially since the numerator is real and positive). The denominator may be written in the form $re^{i\theta}$ where

$$r = \sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2} \quad \text{and} \quad \tan\theta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

Therefore,

$$z_0 = D\frac{\omega_0^2}{r} e^{-i\theta} \equiv A e^{i\varphi}$$

where

$$A = \frac{D}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \left(\frac{2\beta\omega}{\omega_0^2}\right)^2}} \quad \text{and} \quad \tan\varphi = -\frac{2\beta\omega}{\omega_0^2 - \omega^2}$$

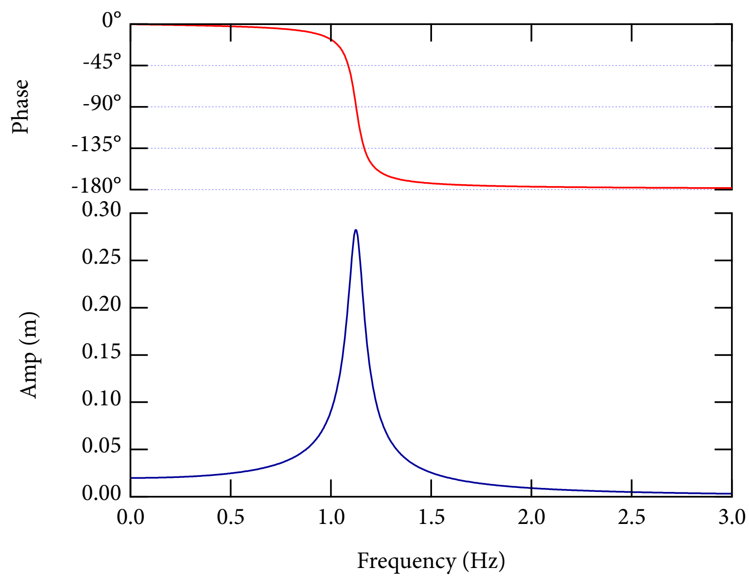


Figure 1: Amplitude A and phase φ vs $f = \omega/2\pi$.

and I have divided both numerator and denominator in A by ω_0^2 to isolate the dimensions entirely in the numerator.

Now it is time to substitute some numerical values:

$$\omega_0 = \sqrt{\frac{10 \text{ N/m}}{0.2 \text{ kg}}} = \sqrt{50} \text{ rad/s}$$

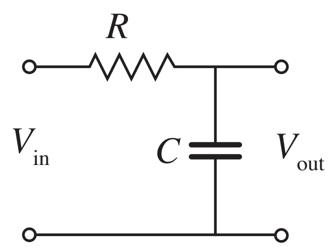
$$\beta = \frac{0.1 \text{ kg/s}}{2 \times 0.2 \text{ kg}} = 0.25 \text{ s}^{-1}$$

Here's the Igor code that produces this graph. I'll post a screencast to illustrate how to make it.

```
Make/N=512 Amp,Phase
SetScale/I x 0,3,"Hz",Amp,Phase
SetScale d 0,0,"°",Phase
SetScale d 0,0,"m",Amp
Amp = (0.02)/sqrt( (1 - (2*pi*x)^2/50)^2 + (2*0.25*2*pi*x/50)^2 )
Phase = atan2( -2*0.25*2*pi*x, 50-(2*pi*x)^2 )*180/pi

Window AmpAndPhase() : Graph
  PauseUpdate; Silent 1   // building window...
  Display /W=(317,89,770,430) Amp
  AppendToGraph/L=phase Phase
  ModifyGraph gFont="Minion Pro",gfSize=14
  ModifyGraph rgb(Amp)=(1,3,39321), grid(phase)=1, tick=2
```

```
  ModifyGraph mirror=1, standoff=0,lblPosMode(phase)=1
  ModifyGraph lblPos(left)=66, linTkLabel(phase)=1
  ModifyGraph axisOnTop(left)=1,axisOnTop(bottom)=1
  ModifyGraph freePos(phase)={0,bottom}
  ModifyGraph axisEnab(left)={0,0.6},axisEnab(phase)={0.65,1}
  ModifyGraph manTick(phase)={0,45,0,0},manMinor(phase)={0,50}
  Label left "Amp (\\U)"
  Label bottom "Frequency (\\U)"
  Label phase "Phase"
  SetAxis/A/N=1 left
  SetAxis/N=1 bottom 0,3
  SetAxis phase -180,0
EndMacro
```



Problem 2 At time $t = 0$ the potential difference V_{in} switches suddenly from 0 to V_0. Calculate V_{out} as a function of time for $t > 0$ and sketch $V_{out}(t)$.

Solution: At $t = 0$, the left end of the resistor goes to V_0 while the right end is still at $V = 0$. The potential difference V_0 across the resistor therefore causes a current $I = V_0/R$ to flow through the resistor, at least instantaneously. As this current deposits charge on the upper plate of the capacitor, a potential difference V_{out} develops across the capacitor, which reduces the potential difference across the resistor, which lowers the current flowing through it. So, the rate at which charge is deposited on the upper plate of the capacitor gradually diminishes to zero as the entire applied voltage V_0 appears across the capacitor.

Algebraically, we have

$$I = \dot{Q} = \frac{V_0 - V_{out}}{R}$$
$$V_{out} = \frac{Q}{C}$$
$$\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC} = \frac{dQ}{dt}$$

We can now separate variables

$$-\frac{dt}{RC} = \frac{dQ}{Q - CV_0}$$

and integrate

$$\begin{aligned}
\int_0^t -\frac{dt'}{RC} &= \int_0^Q \frac{dQ'}{Q' - CV_0} \\
-\frac{t}{RC} &= \ln\left(\frac{Q - CV_0}{-CV_0}\right) = \ln\left(\frac{CV_0 - Q}{CV_0}\right) \\
e^{-t/RC} &= 1 - \frac{Q}{CV_0}
\end{aligned}$$

Since the output voltage is $V_{\text{out}} = Q/C$, we have

$$V_{\text{out}} = V_0 - V_0 e^{-t/RC} = V_0\left(1 - e^{-t/RC}\right)$$

which shows that the voltage across the capacitor approaches V_0 exponentially with a time constant $\tau = RC$. Colloquially, we call this the "RC time" of the circuit. This behavior is illustrated in the graph.