Derivation of the Expression for
Zin for the gyrator circuit of Figure 1
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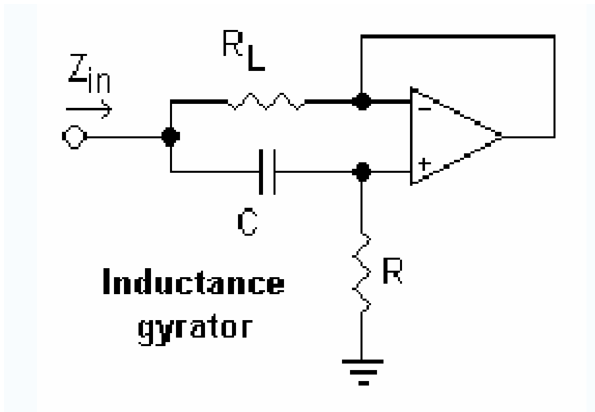


**Figure 1: Gyrator Circuit from Wikipedia.org**

The general expression for Zin is written:

$$Z_{in} = \frac{V_{in}}{I_{in}} \qquad \text{Eq. 1}$$

This search for the expression for Zin of the circuit in figure 1 begins with the derivation of the gain of the op-amp. It should be obvious from inspection that since the output of the op-amp is connected directly to the negative input of the op-amp the following expression is true.

$$e(-) = V_o \qquad \text{Eq. 2}$$

Where e(-) represents the voltage at the op-amp's negative terminal

Using Millman's Theorem, the expressions for the voltage at the positive terminal can be written as shown below.

$$e(+) = \frac{\frac{V_{in}}{X_c}}{\frac{1}{R} + \frac{1}{X_c}} \qquad \text{Eq. 3}$$

Where e(+) represents the voltage at the op-amp's positive terminal

Xc is used to represent the capacitive reactance of the capacitor C in the circuit of figure 1.

$$X_C = \frac{1}{sC} \qquad \text{Eq. 4}$$

Where s represents the complex variable jω

The basic assumption for op-amps indicates that:

$$e(-) = e(+) \qquad \text{Eq. 5}$$

With this assumption in place, it is possible to write:

$$V_o = \frac{\frac{V_{in}}{X_c}}{\frac{1}{R} + \frac{1}{X_c}} \qquad \text{Eq. 6}$$

Multiply the numerator and denominator on the right-hand side of the equation above by *RXc* to arrive at:

$$V_o = \frac{RV_{in}}{R + X_C} \qquad \text{Eq. 7}$$

Next, write the expression for *Iin* from inspection:

$$I_{in} = \frac{V_{in} - V_O}{R_L} + \frac{V_{in}}{R + X_C} \qquad \text{Eq. 8}$$

Expand the first term on the right-hand side of the above expression to arrive at the expression:

$$I_{in} = \frac{V_{in}}{R_L} - \frac{V_O}{R_L} + \frac{V_{in}}{R + X_C} \qquad \text{Eq. 9}$$

Replace *Vo* in the above expression with the expression previously obtained for the gain of the op-amp (Eq. 7).

$$I_{in} = \frac{V_{in}}{R_L} - \frac{V_{in}R}{R_L\left(R + X_C\right)} + \frac{V_{in}}{R + X_C} \qquad \text{Eq. 10}$$

Rearrange the terms on the right-hand side of the above expression to obtain the expression below.

$$I_{in} = \frac{V_{in}}{R_L} + \frac{V_{in}}{R + X_C} - \frac{V_{in}R}{R_L\left(R + X_C\right)} \qquad \text{Eq. 11}$$

Factor *Vin* out of the terms on the right-hand side of the above expression to arrive at:

$$I_{in} = V_{in}\left(\frac{1}{R_L} + \frac{1}{R + X_C} - \frac{R}{R_L\left(R + X_C\right)}\right) \qquad \text{Eq. 12}$$

Dividing both the left-hand and the right-hand side of the expression above by *Vin*, the expression below is obtained. Note that the expression below equates to the input admittance of the gyrator.

$$\frac{I_{in}}{V_{in}} = \left(\frac{1}{R_L} + \frac{1}{R + X_C} - \frac{R}{R_L\left(R + X_C\right)}\right) \qquad \text{Eq. 13}$$

By combining the first and second term of the above expression, the expression below is obtained.

$$\frac{I_{in}}{V_{in}} = \left(\frac{R_L + R + X_C}{R_L\left(R + X_C\right)} - \frac{R}{R_L\left(R + X_C\right)}\right) \qquad \text{Eq. 14}$$

By redistributing the first term on the right-hand side of the above expression, the expression below is arrived at.

$$\frac{I_{in}}{V_{in}} = \left(\frac{R_L + X_C}{R_L\left(R + X_C\right)} + \frac{R}{R_L\left(R + X_C\right)} - \frac{R}{R_L\left(R + X_C\right)}\right) \qquad \text{Eq. 15}$$

The second and third term on the right-hand side of the expression cancel each other to yield.

$$\frac{I_{in}}{V_{in}} = \left(\frac{R_L + X_C}{R_L\left(R + X_C\right)}\right) \qquad \text{Eq. 16}$$

Inverting both the left-hand and right-hand side of the above expression gives:

$$\frac{V_{in}}{I_{in}} = \left( \frac{R_L(R + X_C)}{R_L + X_C} \right) \qquad \text{Eq. 17}$$

Substitute the value for *Xc* to arrive at the following expression:

$$\frac{V_{in}}{I_{in}} = \left( \frac{R_L\left(R + \frac{1}{sC}\right)}{R_L + \frac{1}{sC}} \right) \qquad \text{Eq. 18}$$

Multiply the numerator and denominator in the above expression by *sC* to arrive at the expression:

$$\frac{V_{in}}{I_{in}} = \left( \frac{R_L(sCR + 1)}{sCR_L + 1} \right) \qquad \text{Eq 19}$$

The result of all of this manipulation is the following expression:

$$\frac{V_{in}}{I_{in}} = \left( \frac{sCRR_L + R_L}{sCR_L + 1} \right) \qquad \text{Eq. 20}$$

Next, the expression in Wikipedia will be equated to the result in equation 20 above in effort to establish that they are equivalent.

In Wikipedia, the expression for Zin is written as follows:

$$Z_{IN} = (R_L + sR_L RC) \| \left( R + \frac{1}{sC} \right) \qquad \text{Eq. 21}$$

Equation 20 in this analysis yielded the final expression for Zin as:

$$Z_{IN} = \frac{sCRR_L + R_L}{sCR_L + 1} = \frac{R_L + sR_L RC}{sCR_L + 1} \qquad \text{Eq. 22}$$

The following exercise is undertaken to prove that the two expressions Eq. 21 and Eq. 22 are equivalent.

To begin the proof, the two expressions are tentatively set equal to each other.

$$\left(R_L + sR_L RC\right) \| \left(R + \frac{1}{sC}\right) \overset{?}{=} \frac{R_L + sR_L RC}{sCR_L + 1} \qquad \text{Eq. 23}$$

Expanding the left-hand side of the equation yields the following equality.

$$\frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{R_L + sR_L RC + R + \frac{1}{sC}} \overset{?}{=} \frac{R_L + sR_L RC}{sCR_L + 1} \qquad \text{Eq. 24}$$

The next step involves multiplying the right-hand side of the equation above by the unity factor (R+(1/sC))/(R+(1/sC)) as shown in the expression below. This is done as a means of introducing the term needed to make the numerators on both sides of the equal sign take on the same value.

$$\frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{R_L + sR_L RC + R + \frac{1}{sC}} \overset{?}{=} \left(\frac{R_L + sR_L RC}{sCR_L + 1}\right)\left(\frac{R + \frac{1}{sC}}{R + \frac{1}{sC}}\right) \qquad \text{Eq. 25}$$

Next, the right-hand side of the expression above is collected into a single fractional expression.

$$\frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{R_L + sR_L RC + R + \frac{1}{sC}} \overset{?}{=} \frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{\left(sCR_L + 1\right)\left(R + \frac{1}{sC}\right)} \qquad \text{Eq. 26}$$

By expanding the expression in the right-hand side denominator, the overall expression takes on the form below.

$$\frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{R_L + sR_L RC + R + \frac{1}{sC}} \overset{?}{=} \frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{sCR_L R + R + R_L + \frac{1}{sC}} \qquad \text{Eq. 27}$$

Some rearranging of the terms in the right-hand side denominator yields the final answer which appears to indicate that the two expressions are indeed equal.

$$\frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{R_L + sR_L RC + R + \frac{1}{sC}} = \frac{\left(R_L + sR_L RC\right)\left(R + \frac{1}{sC}\right)}{R_L + sR_L RC + R + \frac{1}{sC}} \qquad \text{Eq. 28}$$

With the final expression in Eq. 28, it is fairly safe to conclude that the simplified expression for Zin contained in Eq. 20 describes the impedance of the gyrator from the Wikipedia presentation and pictured in Figure 1 at the beginning of this document.