The Associated Legendre functions are related to the ordinary Legendre polynomials by the following formula:

$$P_l^m(x) = (-1)^m \left(1 - x^2\right)^{m/2} \frac{d^m}{dx^m} P_l(x)$$

Upon substitution into this formula, we obtain,

$$P_l^m(x) = \frac{(-1)^m}{2^l\, l!} \left(1 - x^2\right)^{m/2} \frac{d^{l+m}}{dx^{l+m}} \left(1 - x^2\right)^l$$

The spherical harmonics are defined in terms of these functions as:

$$Y_{lm}(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\; P_l^m(\cos\theta)\, e^{im\phi}$$

The first few harmonics are:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi} \qquad Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_{30} = \sqrt{\frac{7}{4\pi}}\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) \qquad Y_{31} = -\frac{1}{4}\sqrt{\frac{21}{4\pi}}\sin\theta\left(5\cos^2\theta - 1\right) e^{i\phi}$$

$$Y_{32} = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\sin^2\theta\cos\theta\, e^{2i\phi} \qquad Y_{33} = -\frac{1}{4}\sqrt{\frac{35}{4\pi}}\sin^3\theta\, e^{3i\phi}$$

The spherical harmonics are *normalized*, so that,

$$\int_0^{2\pi}\int_0^{\pi} Y_{lm}^* Y_{pq} \sin\theta\, d\theta = \delta_{lp}\delta_{mq}$$