Damped Harmonic Oscillator

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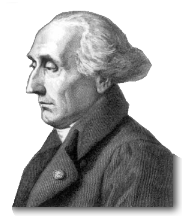
A simple harmonic oscillator subject to linear damping may oscillate with exponential decay, or it may decay biexponentially without oscillating, or it may decay most rapidly when it is critically damped. When driven sinusoidally, it resonates at a frequency near the natural frequency, and with very large amplitude when the damping is slight. Because the system is linear, we can superpose solutions, leading to Green's method and the usefulness of Laplace transforms.

Physics 111



Many systems exhibit mechanical stability: disturbed from an equilibrium position they move back toward that equilibrium position. This is the recipe for oscillation. Provided that the disturbance from equilibrium is small, virtually any mechanically stable system will experience a linear restoring force, giving rise to simple harmonic motion.

We will start with a single degree of freedom, which will illustrate most of the important behavior: decaying oscillation and resonance. We will soon generalize to a system with an arbitrary number of degrees of freedom, where we will find that we can always find suitable combinations of coordinates to reduce that problem to a set of decoupled one-dimensional oscillators. The simple harmonic oscillator model, therefore, is ubiquitous in physics. You find it in mechanics; in electromagnetism, where it describes electromagnetic waves, plasmon resonances, and laser modes; atomic physics, where it describes coupling of an atom to the electromagnetic field…I suspect it arises in every subdiscipline of physics. In fact, one of my graduate physics professors quipped that it couldn't be a physics course with the simple harmonic oscillator.

Our point of departure is the general form of the lagrangian of a system near its position of stable equilibrium, from which we deduce the equation of motion. We will then consider both unforced and periodically forced motion before turning to general methods of solving the linear second-order differential equations that describe oscillatory systems.

1. General Form of the Lagrangian

We may Taylor expand the potential about $q = 0$:

$$U(q) = U(0) + \left.\frac{dU}{dq}\right|_0 q + \frac{1}{2}\left.\frac{d^2U}{dq^2}\right|_0 q^2 + \cdots \tag{1}$$

The first term on the right-hand side is just a constant; it cannot influence the dynamics since the variational derivative kills all constants. The second term vanishes since we expand about a minimum. The third term has to be positive (otherwise, we expand about either a maximum or a saddle point). So,

$$U(q) = U(0) + \frac{1}{2}\frac{d^2U}{dq^2}q^2 + \cdots = U_0 + \frac{1}{2}kq^2 + \cdots \tag{2}$$

We may also expand the kinetic energy about its equilibrium value, 0:

$$T = \frac{1}{2} m \dot{q}^2 \tag{3}$$

for some constant m.

Lagrange's equations for this system then become

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} \\ &= \left(\frac{\partial^2 L}{\partial \dot{q}^2}\right)\ddot{q} - \left(\frac{\partial^2 L}{\partial q^2}\right) q \\ 0 &= \ddot{q} + \frac{k}{m} q \end{aligned} \tag{4}$$

If $k = \frac{\partial^2 U}{\partial q^2} = -\frac{\partial^2 L}{\partial q^2}$ is negative, then this gives exponential solutions and the equilibrium is unstable. If it is positive, then the equilibrium is stable and the system oscillates about the equilibrium position $q = 0$. In that case, Eq. (4) is the simple harmonic oscillator (SHO) equation, with solutions of the form

$$q(t) = A\cos\left(\omega_0 t + \varphi\right) = A \operatorname{Re} e^{-i(\omega_0 t + \varphi)}$$

where $\omega_0 \equiv \sqrt{k/m}$ is called the **natural frequency** of the oscillator and the coefficients A and φ may be determined from initial conditions.

I am using $-i$ in the exponent to be consistent with quantum mechanics. A plane wave of the form $e^{i(kx-\omega t)}$ moves to positive x if k is positive. Furthermore, the momentum of the wave is $\hbar k$, not $-\hbar k$, which is possible because the time component carries the negative sign.

There is another way to interpret the complex expression. Moving the real number A inside and rearranging gives

$$q(t) = \operatorname{Re}\left[\left(A e^{-i\varphi}\right) e^{-i\omega_0 t}\right] = \operatorname{Re}\left[\mathbb{A} e^{-i\omega_0 t}\right]$$

where $\mathbb{A} = A e^{-i\phi}$ is the complex amplitude of oscillation. The single complex amplitude contains both the magnitude and phase information of the oscillation.

2. Damped Simple Harmonic Oscillator

If the system is subject to a *linear* damping force, $\mathbf{F} = -b\dot{\mathbf{r}}$ (or more generally, $-b_j \dot{\mathbf{r}}_j$), such as might be supplied by a viscous fluid, then Lagrange's equations must be modified to include this force, which cannot be derived from a potential. Recall that we had developed the expression

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \mathbf{F}^{\text{ext}} \cdot \frac{\partial \mathbf{r}}{\partial q} \tag{5}$$

If the equations of transformation do not depend explicitly on the time—so that $\partial \mathbf{r}_j / \partial q = \partial \dot{\mathbf{r}}_j / \partial \dot{q}$—then we can simplify the damping term:

$$-\sum_j b_j \dot{\mathbf{r}}_j \frac{\partial \mathbf{r}_j}{\partial q} = -\sum_j b_j \dot{\mathbf{r}}_j \frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}} = -\frac{\partial}{\partial \dot{q}} \sum_j \frac{b_j}{2} (v_j)^2 \equiv -\frac{\partial \mathsf{F}}{\partial \dot{q}}$$

The sum in this final expression, which can be generalized to N particles in the obvious way, is called **Rayleigh's dissipation function**, F. The modified Lagrange equations are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial \mathsf{F}}{\partial \dot{q}} = 0 \tag{6}$$

Rayleigh's dissipation function allows a general linear damping term to be incorporated into the Lagrangian formalism.

For a single degree of freedom with linear damping, we have $\mathsf{F} = \frac{b}{2}\dot{q}^2$, so

$$m\ddot{q} + kq + b\dot{q} = 0 \qquad \text{or} \qquad \ddot{q} + 2\beta\dot{q} + \omega_0^2 q = 0 \tag{7}$$

where we have defined $\beta = b/2m$, which has the dimensions of a frequency, and is called the **damping parameter**. We may solve this equation with the *Ansatz* $q = e^{st}$, which slightly biases us to expect exponential solutions, or with the *Ansatz* $q = e^{-i\omega t}$, which is more appropriate when we expect oscillatory solutions. Making the latter choice yields the algebraic equation

$$-\omega^2 q - 2i\beta\omega q + \omega_0^2 q = 0 \qquad \Longrightarrow \qquad \omega = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

This yields two linearly independent solutions to the second-order differential equation, Eq. (11), unless $\beta = \omega_0$. This special case is called **critical damping**; the second solution takes the form $te^{-\beta t}$ when $\beta = \omega_0$.

When $\beta < \omega_0$, the oscillator is **underdamped** and oscillates when released from rest away from its equilibrium position. Let

$$\boxed{\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}} \tag{8}$$

An **underdamped** oscillator vibrates at angular frequency ω_1 with decreasing amplitude when released away from its equilibrium position. For light damping, this frequency is just slightly smaller than the natural frequency ω_0.

Then the solution may be expressed

$$q(t) = Ae^{-\beta t}\cos(\omega_1 t + \varphi) \tag{9}$$

When $\beta > \omega_0$, the oscillator is **overdamped**; the two solutions decay exponentially with different time constants:

$$q(t) = A_+ e^{-\left(\beta + \sqrt{\beta^2 - \omega_0^2}\right)t} + A_- e^{-\left(\beta - \sqrt{\beta^2 - \omega_0^2}\right)t} \tag{10}$$

The second term has the longer time constant; it tends to dominate at long times.

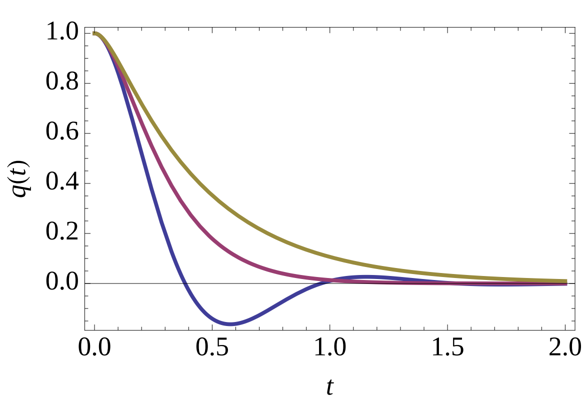


Figure 1: *Overdamped (olive), critically damped (purple), and underdamped (blue) harmonic oscillators released from rest.*

3. Driven DSHO

A damped simple harmonic oscillator subject to a sinusoidal driving force of angular frequency ω will eventually achieve a steady-state motion at the same frequency ω. How long it must be driven before achieving steady state depends on the damping; for very light damping it can take a great many cycles before the transient solution to the homogeneous differential equation decays sensibly to zero, as illustrated in Fig. 2.

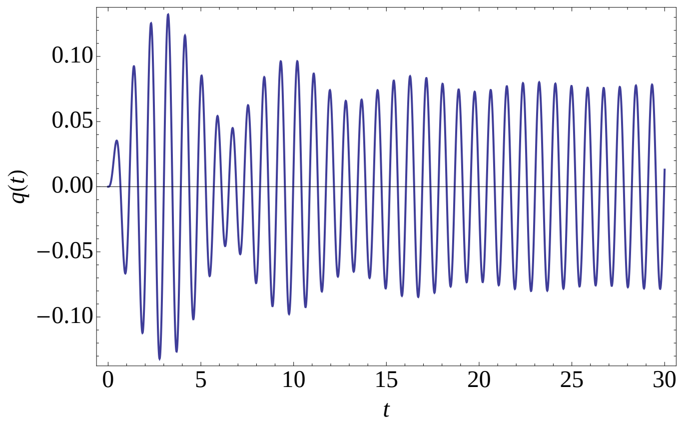


Figure 2: *Response of a lightly damped simple harmonic oscillator driven from rest at its equilibrium position. In this case, $\omega_0/2\beta \approx 20$ and the drive frequency is 15% greater than the undamped natural frequency.*

The equation of motion for the driven damped oscillator is

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{F_0}{m}\cos\omega t = \text{Re}\left(\frac{F_0}{m}e^{-i\omega t}\right) \tag{11}$$

Rather than solving the problem for the sinusoidal forcing function, let us instead look for a complex function of time, $z(t)$, that satisfies essentially the same equation,

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{-i\omega t} \tag{12}$$

If we can find such a function, then its real part $q = \text{Re}\, z$ solves Eq. (11). Knowing that the solution will eventually oscillate at the same frequency as the drive, we make the *Ansatz* $z = \mathbb{A}e^{-i\omega t}$, thereby obtaining the particular solution

$$\mathbb{A} = \frac{F_0/m}{\omega_0^2 - \omega^2 - 2\beta\omega i} \qquad \text{or} \qquad q = \text{Re}\left[\frac{F_0/m}{\omega_0^2 - \omega^2 - 2\beta\omega i}e^{-i\omega t}\right] \tag{13}$$

where the complex amplitude $\mathbb{A}$ encodes both the (real) amplitude A and the phase of the oscillator with respect to the drive, $\mathbb{A} = Ae^{-i\varphi}$, and

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}} \qquad \text{and} \qquad \varphi = \tan^{-1}\left(\frac{2\omega\beta}{\omega^2 - \omega_0^2}\right)$$

The complete solution is then the sum of the general solution, Eq. (9), and the particular solution Eq. (13).

Exercise 1 Show that the frequency of maximum amplitude (the resonant frequency) is $\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$.

The width of the resonance may be characterized by the **quality factor** of the resonator, a dimensionless quantity that is defined by

Roughly speaking, the quality of a resonator is the number of oscillations it undergoes before its initial energy is reduced by a factor of $1/e$.

$$Q \equiv \frac{\omega_R}{2\beta} = \frac{\sqrt{\omega_0^2 - 2\beta^2}}{2\beta} \tag{14}$$

Figure 3 shows resonance curves for damped driven harmonic oscillators of several values of Q between 1 and 256. For a lightly damped oscillator, you can show that $Q \approx \frac{\omega_0}{\Delta\omega}$, where $\Delta\omega$ is the frequency interval between the points that are down to $1/\sqrt{2}$ from the maximum amplitude. The width is really defined with respect to the energy of the oscillator, which is proportional to the square of the amplitude. Hence, the factor of $1/\sqrt{2}$.

Exercise 2 If a tuning fork oscillating at 440 Hz takes 4 seconds to lose half its energy, roughly what is its quality? *Ans:* ≈ 2500.

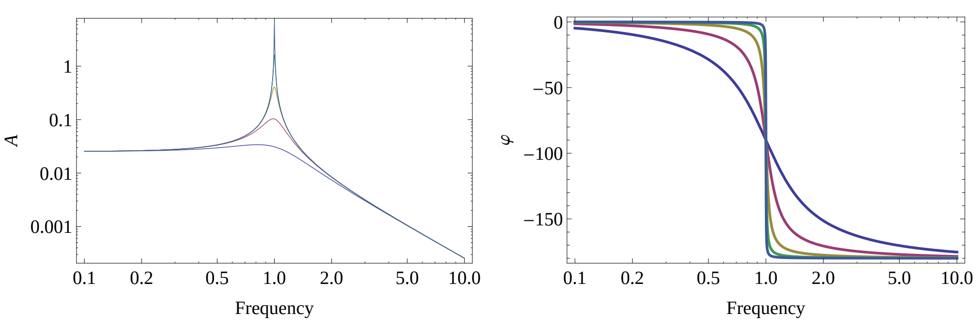


Figure 3: *Amplitude and phase of a damped driven simple harmonic oscillator for several different quality factors (Q = 1, 4, 16, 64, and 256).*

4. Green's Function

If the damped oscillator is driven by an arbitrary function of time,

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = \frac{F(t)}{m} \tag{15}$$

there are a variety of ways to solve for $q(t)$. All are based on the observation that the left-hand side of this equation is a **linear operator** on q, $\mathbf{L}(q)$. That is,

$$\mathbf{L} = \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

Linear operators satisfy

$$\mathbf{L}(q_1 + q_2) = \mathbf{L}(q_1) + \mathbf{L}(q_2)$$
$$\mathbf{L}(\alpha_1 q_1 + \alpha_2 q_2) = \alpha_1\mathbf{L}(q_1) + \alpha_2\mathbf{L}(q_2)$$

This means that if we have a sum of forcing functions, $F_i(t) = m f_i(t)$, then

$$\mathbf{L}\left(\sum_i \alpha_i q_i(t)\right) = \sum_i \mathbf{L}\left(\alpha_i q_i(t)\right) = \sum_i \alpha_i f_i(t)$$

which is the principle of superposition. It says that we are free to break apart the forcing function into pieces, solve each piece separately, and add up the resulting solutions.

George Green (1793–1841) looked for a solution to Eq. (15) by breaking apart the forcing function into a series of brief impulses, short time intervals Δt of constant force:

$$\mathbf{L}(q(t)) = \begin{cases} 0 & t < t_0 + j\Delta t \\ f_j & t_0 + j\Delta t < t < t_0 + (j+1)\Delta t \\ 0 & t > t_0 + (j+1)\Delta t \end{cases}$$

The complete temporal behavior of q is then obtained by adding up all the individual solutions $q_j(t)$ corresponding to all the little impulses f_j:

$$q(t) = \sum_j q_j(t)$$

In the limit that Δt goes to zero, we may represent the forcing functions by $f_j(t') = f_j\,\delta(t' - t_j)$, where $\delta(t)$ is Dirac's delta function. Recall that the delta function $\delta(x)$ has the properties

- $\delta(x) = 0$ if $x \neq 0$
- $\delta(0) \to \infty$
- $\int \delta(x)\,dx = 1$, provided that the range of integration includes $x = 0$, and vanishes otherwise.

Think of $\delta(x)$ as an infinitely tall spike at $x = 0$ with unit area. Using the delta function, we can represent the forcing function via

$$f(t) = \int_{-\infty}^{\infty} f(t')\delta(t - t')\,dt'$$

where t' is a dummy variable of integration. The only time that the delta function is nonzero is when $t = t'$. Integrating over this time yields $f(t)$ times the integral of the delta function, which is 1 by definition. So, if we knew the solution to

$$\ddot{G} + 2\beta\dot{G} + \omega_0^2 G = \delta(t - t') \tag{16}$$

for a function $G(t, t')$, we could multiply $G(t, t')$ by $f(t')$ and integrate over all times $t' < t$ to get the solution at t:

$$\boxed{q(t) = \int_{-\infty}^{t} G(t, t') f(t')\, dt'} \tag{17}$$

where $f(t') = F(t')/m$.

Exercise 3 Show that

$$G(t, t') = \begin{cases} 0 & t < t' \\ \frac{1}{\omega_1} e^{-\beta(t-t')} \sin\omega_1(t - t') & t \geq t' \end{cases} \tag{18}$$

Hint: Solve for $t \neq t'$ and then integrate the defining differential equation from $t' - \epsilon$ to $t' + \epsilon$ and take the limit as $\epsilon -> 0$. Then match the solution to $G(t, t') = 0$ for $t < t'$.

5. Laplace Transform

Another important technique for solving Eq. (15) is via an integral transform called the **Laplace transform**. The transform converts time derivatives into polynomials, which produces an algebraic equation. These are easier to solve than differential equations. After solving the algebraic equation, we then apply the inverse Laplace transform to return to a time-domain expression that gives $q(t)$.

First, we define the Laplace transform of a function of time, $f(t)$, as

$$\mathscr{F}(s) = \mathscr{L}\{f(t)\} \equiv \int_0^\infty e^{-st} f(t)\,dt \tag{19}$$

Since we will be applying this to time derivatives of generalized coordinates, let's work out expressions for the Laplace transform of df/dt and d^2f/dt^2:

$$\mathscr{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty e^{-st}\frac{df}{dt}\,dt = e^{-st}f\Big|_0^\infty + \int_0^\infty se^{-st}f\,dt = s\mathscr{F}(s) - f(0) \tag{20}$$

$$\mathscr{L}\left\{\frac{d^2f}{dt^2}\right\} = \int_0^\infty e^{-st}\frac{d^2f}{dt^2}\,dt = s^2\mathscr{F}(s) - sf(0) - f'(0) \tag{21}$$

where the prime indicates differentiation with respect to the argument of the function.

We now seek to apply the Laplace transform to Eq. (15). Let us assume that we have situated the origin of time such that the system is quiescent and the forcing function vanishes for $t < 0$. Then we may take all of the integrated terms to vanish, and get

$$(s^2 + 2\beta s + \omega_0^2)\mathscr{Q}(s) = \mathscr{L}\left\{\frac{F(t)}{m}\right\} = \frac{1}{m}\mathscr{F}(s) \tag{22}$$

Solving for the Laplace transform of q, $\mathscr{Q}(s)$, gives

$$\mathscr{Q}(s) = \frac{\mathscr{F}(s)/m}{s^2 + 2\beta s + \omega_0^2} \tag{23}$$

We now apply the Laplace transform inversion integral (the **Bromwich integral**),

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \mathscr{Q}(s)e^{st}\,ds \tag{24}$$

where γ is a real constant that exceeds the real part of all the singularities of $\mathscr{F}(s)$, to solve for $q(t)$. Equation (24) is derived in the notes on contour integration.

Example 1 Suppose that the oscillator described by Eq. (11) is thumped at $t = 0$ with a delta-function impulse: $F(t) = \alpha\delta(t)$. Find $q(t)$ using the Laplace transform method.

According to Eq. (23), we need first to calculate the Laplace transform of the forcing function:

$$\mathscr{L}\{\alpha\delta(t)\} = \int_0^\infty e^{-st}\alpha\delta(t)\,dt = \alpha$$

By Eq. (23) and the inversion integral, we have

$$q(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\alpha/m}{s^2+2\beta s+\omega_0^2}e^{st}\,ds$$

The integrand has poles at the roots of the quadratic equation in the denominator, $s_\pm = -\beta \pm \sqrt{\beta^2-\omega_0^2}$, both of which lie to the left of $s = 0$. So, we may integrate along the imaginary axis and close in the left half-plane, where the exponential sends the integrand to zero for $t > 0$. By the residue theorem, the integral is therefore $2\pi i$ times the sum of the two residues. Writing the denominator as $(s-s_+)(s-s_-)$, the residues are

$$a_{-1}^{+} = \frac{\alpha/m}{s_+ - s_-}e^{s_+ t} \quad \text{and} \quad a_{-1}^{-} = \frac{\alpha/m}{s_- - s_+}e^{s_- t}$$

Combining these results gives

$$q(t) = \frac{\alpha}{m}\frac{e^{-\beta t}}{2\sqrt{\beta^2-\omega_0^2}}\left(e^{\sqrt{\beta^2-\omega_0^2}\,t} - e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right)$$

Noting that we have called $\omega_1 = \sqrt{\omega_0^2-\beta^2}$, we may rewrite this results as

$$q(t) = \frac{\alpha}{m}\frac{e^{-\beta t}}{i\omega_1}\frac{e^{i\omega_1 t}-e^{-i\omega_1 t}}{2i} = \frac{\alpha e^{-\beta t}}{m\omega_1}\sin(\omega_1 t)$$

consistent with what we found before.